Matematisk-fysiske Meddelelser
udgivet af
Det Kongelige Danske Videnskabernes Selskab
Bind **31**, nr. 14

Mat. Fys. Medd. Dan. Vid. Selsk. **31**, no. 14 (1959)

THE ENERGY-MOMENTUM COMPLEX IN THE GENERAL THEORY OF RELATIVITY

BY

C. MØLLER

København 1959
i kommission hos Ejnar Munksgaard

CONTENTS

Synopsis

It is shown that the expression for the complex of energy-momentum derived in an earlier paper from physical arguments also follows directly from the mathematical invariance properties of the theory. The usual method of infinitesimal coordinate transformations is generalized to the case of a variational principle where the integrant of the integral to be varied depends on the derivatives of the field variables of arbitrarily high order. The method is then applied separately to the gravitational field and the matter field. The transformation properties of the complex under arbitrary space-time transformations are derived, and a closer specification of the notion of "local systems of inertia" is given.

Printed in Denmark
Bianco Lunos Bogtrykkeri A-S

1. Introduction and Summary

In a generally covariant theory like EINSTEIN's theory of gravitation, where the field equations are derivable from a variational principle, it is possible to define a large number of quantities which are "conserved" (1). Therefore, extra criteria are needed in order to select out of this multitude of conserved quantities those which have a physical meaning. In particular, it becomes a problem to find the correct expressions for the pseudo-tensor of energy and momentum. For a Langrangean system where the field equations are derivable from a Langrangean density $\mathfrak{L}$, which is a function of the field variables and their first order derivatives only and which transforms like a scalar density under arbitrary space-time transformations, the well-known "method of infinitesimal transformations" leads to a natural choice of the energy-momentum complex.*

In the case of gravitational fields, the field equations may now be written in the Langrangean form with a Langrangean density $\mathfrak{L} = \mathfrak{L}(g^{ik}, g_l^{ik})$ which is a scalar density only under arbitrary *linear* transformations. Therefore, in applying the method of infinitesimal transformations, one is restricted to linear transformations, and the "canonical" energy-momentum complex $\Theta_i^{\ k}$ obtained in this way does not possess all the transformation properties required for a physical interpretation of its components. The canonical complex $\Theta_i^{\ k}$ following from the invariance of $\mathfrak{L}$ under arbitrary infinitesimal linear transformations is of the form

$$\Theta_i^{\ k} = \sqrt{-g}\,(T_i^k + \vartheta_i^{\ k}) = s_i^{\ kl}{}_{,l}. \tag{1}$$

Here, T_i^k is the matter tensor which appears on the right-hand side of the gravitational field equations

$$G_{ik} \equiv R_{ik} - \frac{1}{2} g_{ik} R = -\varkappa\, T_{ik}. \tag{2}$$

* We adopt the terminology of LORENTZ who used the denotation *complex* for a covariant quantity with tensor indices which, however, behaves like a tensor or tensor density under *linear* space-time transformations only.

$g = \det\{g_{ik}\}$ is the determinant of the metric tensor g_{ik} and

$$\sqrt{-g}\,\vartheta_i^{\ k} = \frac{1}{2\varkappa}\left\{\frac{\partial \mathfrak{L}}{\partial g_k^{lm}}\, g_i^{lm} - \delta_i^k\, \mathfrak{L}\right\}. \tag{3}$$

Further,

$$s_i^{\ kl} = \frac{1}{\varkappa}\frac{\partial \mathfrak{L}}{\partial g_l^{im}}\, g^{km} \tag{4}$$

is the quantity introduced by EINSTEIN and by TOLMAN[2], and $\Theta_i^{\ k}$ satisfies the divergence relation

$$\Theta_{i\ ,\,k}^{\ k} \equiv \frac{\partial\, \Theta_i^{\ k}}{\partial x^k} = 0\,. \tag{5}$$

A simple calculation shows[3] that $s_i^{\ kl}$ is of the form

$$s_i^{\ kl} = h_i^{\ kl} + a_{i\ \ ,\,m}^{\ klm} \tag{6}$$

where

$$\left.\begin{aligned} h_i^{\ kl} &= -h_i^{\ lk} = \frac{g_{in}}{2\varkappa\sqrt{-g}}\left[(-g)\,(g^{kn}\,g^{lm} - g^{ln}\,g^{km})\right]_{,m} \\ a_i^{\ klm} &= -a_i^{\ kml} = \frac{\sqrt{-g}}{2\varkappa}\,(\delta_i^l\, g^{km} - \delta_i^m\, g^{kl})\,. \end{aligned}\right\} \tag{7}$$

On account of the antisymmetry of the last quantity in l and m, $a_{i\ \ ,\,l,\,m}^{\ klm}$ is zero and, by (1) and (6), $\Theta_i^{\ k}$ may be expressed in terms of the "super-potentials" $h_i^{\ kl}$ as

$$\Theta_i^{\ k} = h_{i\ \ ,\,l}^{\ kl}. \tag{8}$$

Now, since $h_i^{\ kl}$ is antisymmetric in k and l, the relation (5) is a simple consequence of (8).

Although the integrals

$$P_i = \frac{1}{c}\int \Theta_i^{\ 4}\, dx^1\, dx^2\, dx^3 \tag{9}$$

give correct values for the total energy and momentum of a closed system, at least if one applies quasi-Galilean coordinates, $\Theta_i^{\ k}$ is not the correct expression for the complex of energy and momentum, since it fails in a physically meaningful manner to account for the distribution of the energy and the energy current in space. In a recent paper in the Annals of Physics [4], a different expression for the energy-momentum complex was proposed. It is defined by

$$\mathcal{T}_i^{\,k} = \sqrt{-g}\,(T_i^{\,k} + t_i^{\,k}) \tag{10}$$

$$\sqrt{-g}\, t_i^{\,k} = \sqrt{-g}\left(-\frac{1}{\varkappa} G_i^k + 2\,\vartheta_i^{\,k}\right) - \delta_i^k\, h_r^{\,rl}{}_{,l} + h_r^{\,rk}{}_{,i}\,. \tag{11}$$

If T_i^k is eliminated by means of the field equations (2), $\mathcal{T}_i^{\,k}$ may be expressed in terms of a superpotential $\chi_i^{\,kl}$ as

$$\mathcal{T}_i^{\,k} = \chi_i^{\,kl}{}_{,l} \tag{12}$$

with

$$\chi_i^{\,kl} = -\chi_i^{\,lk} = 2\,h_i^{\,kl} - \delta_i^{\,k}\, h_r^{\,rl} + \delta_i^{\,l}\, h_r^{\,rk} = \frac{\sqrt{-g}}{\varkappa}(g_{in,\,m} - g_{im,\,n})\, g^{km}\, g^{ln}, \tag{13}$$

and thus

$$\mathcal{T}_i^{\,k}{}_{,k} = 0\,. \tag{13'}$$

For a closed system, the complex $\mathcal{T}_i^{\,k}$ gives the same values for the total momentum and energy

$$P_i = \frac{1}{c}\int \mathcal{T}_i^{\,4}\, dx^1\, dx^2\, dx^3 \tag{14}$$

as the canonical quantity $\Theta_i^{\,k}$ in (9), at least in cases where the latter expressions give meaningful results at all. But the expressions (14) are more general and give correct values for the energy also when the integration is extended over finite regions of space. This is connected with the fact that $\mathcal{T}_4^{\,k}$ in contrast to $\Theta_4^{\,k}$, transforms like a vector density under arbitrary purely spatial transformations

$$\bar{x}^\iota = f^\iota(x^\varkappa), \quad \bar{x}^4 = x^4, \tag{15}$$

a property which is a necessary condition for the possibility of interpreting $\mathcal{T}_4^{\,4}$ and $\mathcal{T}_4^{\,\varkappa}$ as densities of energy and energy current, respectively. Moreover, in later papers [5], it was shown that the pseudo-tensor density $\mathcal{T}_i^{\,k}$ defined by (10) – (13) is uniquely determined by this requirement. From a physical point of view, it would therefore seem that $\mathcal{T}_i^{\,k}$ is the correct expression for the complex of energy and momentum, but the fact remains that the method of infinitesimal transformations applied to the Lagrangean $\mathfrak{L}$ leads to the expression $\Theta_i^{\ k}$ which speaks in favour of the canonical quantity $\Theta_i^{\ k}$.

In the present paper we shall see, however, that the method of infinitesimal space-time transformations leads exactly to the complex $\mathcal{T}_i^{\ k}$ if one starts from another form of the variational principle. It is well known that the gravitational field equations may be obtained from a non-Lagrangean variational principle where the integrant of the integral to be varied is the curvature scalar density

$$\mathfrak{R} = \sqrt{-g}\, R \tag{16}$$

which is a function of the g^{ik} and their first *and* second order space-time derivatives. In fact, this variational principle is usually the starting point in the derivation of the Lagrangean principle. In contrast to the Lagrangean $\mathfrak{L}$, $\mathfrak{R}$ is a scalar density under *arbitrary* space-time transformations. The method of infinitesimal transformations applied to $\mathfrak{R}$ instead of $\mathfrak{L}$ therefore leads to a complex with more extended invariance properties and, as we shall see in section **3**, it just leads to the quantity $\mathcal{T}_i^{\ k}$.

In section **2**, the "method of infinitesimal transformations" is described in the general case of field equations derivable from a non-Lagrangean variational principle where the integrant V in the variation integral depends also on derivatives of the field variables of higher than the first order. In section **3**, the method is applied to the gravitational field, where V is equal to $\mathfrak{R}/\varkappa$. As mentioned above, this leads directly to the relations (10)–(13). As an illustration, we also apply the method to the matter Lagrangean density in which case of course the well-known results of Rosenfeld and of Belinfante[(6)] regarding the symmetrical form of the matter energy tensor are obtained. This is shown in section **4**. In the remaining sections, the transformation properties of $\mathcal{T}_i^{\ k}$ under arbitrary space-time transformations are investigated in some detail. The results obtained suggest a specification of the notion of a local system of inertia.

2. The Method of Infinitesimal Transformations for a Non-Lagrangean System of Fields

Consider a generally non-closed system of fields with the field variables $Y^A(x)$ and their space-time derivatives

$$Y^A_i \equiv \frac{\partial Y^A}{\partial x^i}, \quad Y^A_{i,k} \equiv \frac{\partial^2 Y^A}{\partial x^i \partial x^k}, \ldots,$$

the field equations of which are derivable from a variational principle. Let us first assume that the integrand V in the variation integral is an algebraic function of the Y^A and their first and second order derivatives only. The field equations will then be of the form

$$\frac{\delta V}{\delta Y^A} = -J^A, \tag{17}$$

where the J^A are the "sources" of the field depending in general also on variables other than the Y^A and their derivatives. Further, the

$$\frac{\delta V}{\delta Y^A} = \frac{\partial V}{\partial Y^A} - \left(\frac{\partial V}{\partial Y^A_i}\right)_{,i} + \left(\frac{\partial V}{\partial Y^A_{i,k}}\right)_{,i,k} \tag{18}$$

are the "variational derivatives" of V with respect to Y^A. The partial derivatives occurring in (18) are of a somewhat symbolic character, since the Y^A, Y^A_i, $Y^A_{i,k}$ are not in general truly independent variables. They are defined in the following way. Consider arbitrary variations δY^A of the Y^A, which imply definite variations

$$\delta Y^A_i = (\delta Y^A)_{,i}, \quad \delta Y^A_{i,k} = (\delta Y^A)_{,i,k} \tag{19}$$

of the Y^A_i and $Y^A_{i,k}$ as well as of the algebraic expression $V(Y^A, Y^A_i, Y^A_{i,k})$. The partial derivatives in (18) are now *defined* as the coefficients of δY^A, δY^A_i, and $\delta Y^A_{i,k}$, respectively, in the variation δV of V, i. e.

$$\delta V = \frac{\partial V}{\partial Y^A}\delta Y^A + \frac{\partial V}{\partial Y^A_i}\delta Y^A_i + \frac{\partial V}{\partial Y^A_{i,k}}\delta Y^A_{i,k} \tag{20}$$

(summation over A, i and k!)

Since $\delta Y^A_{i,k} = \delta Y^A_{k,i}$, we can arrange the terms in (20) such as to make the coefficients of $\delta Y^A_{i,k}$ and $\delta Y^A_{k,i}$ equal. With this convention we have

$$\frac{\partial V}{\partial Y^A_{i,k}} = \frac{\partial V}{\partial Y^A_{k,i}}. \tag{21}$$

If the variables Y^A are not independent, as in some of the later applications, say if $Y^A = Y^{A'}$, we use a similar symmetrization rule so as to make

$$\frac{\partial V}{\partial Y^A} = \frac{\partial Y}{\partial Y^{A'}}. \tag{22}$$

Now, consider an arbitrary infinitesimal space-time transformation

$$\overline{x}^i = x^i + \xi^i(x). \tag{23}$$

In all cases considered in the following, the local variation

$$\delta Y^A = \overline{Y}^A(x) - Y^A(x) \tag{24}$$

of $Y^A(x)$ is of the form

$$\delta Y^A = u^{Ak}_{i}\, \xi^i_{,k} - Y^A_i\, \xi^i, \tag{25}$$

where the u^{Ak}_i are linear functions of the field variables. Hence, by (19),

$$\delta Y^A_l = u^{Ak}_{i}\, \xi^i_{k,l} + (u^{Ak}_{i,l} - Y^A_i\, \delta^k_l)\, \xi^i_{,k} - Y^A_{i,l}\, \xi^i. \tag{26}$$

In order to assure general covariance of the field equations (17) we shall now assume that V is a scalar density. Therefore, we must have

$$\delta V + (V \xi^k)_{,k} = 0 \tag{27}$$

at every point in 4-space and for arbitrary functions $\xi^i(x)$. If we integrate (27) over a finite region Ω in 4-space, we get by partial integrations for all functions $\xi^i(x)$ which vanish, together with their first and second order derivatives, at the boundary surface of Ω

$$\int \delta V\, dx = \int \frac{\delta V}{\delta Y^A}\, \delta Y^A\, dx = 0, \tag{28}$$

where $\dfrac{\delta V}{\delta Y^A}$ is the variational derivative defined by (18).

Hence, by (25), after a further partial integration,

$$-\int \left[\frac{\delta V}{\delta Y^A}\, Y^A_i + \left(\frac{\delta V}{\delta Y^A}\, u^{Ak}_i \right)_{,k} \right] \xi^i\, dx = 0. \tag{29}$$

Since the functions $\xi^i(x)$ can be chosen arbitrarily inside Ω, we must have the identity

$$\frac{\delta V}{\delta Y^A} Y^A_i + \left(\frac{\delta V}{\delta Y^A} u^{Ak}_i\right)_{,k} = 0. \tag{30}$$

The expression (20) for the variation of V may also be written

$$\delta V = \frac{\delta V}{\delta Y^A} \delta Y^A + \left[\left[\frac{\partial V}{\partial Y^A_k} - \left(\frac{\partial V}{\partial Y^A_{k,l}}\right)_{,l}\right] \delta Y^A + \frac{\partial V}{\partial Y^A_{k,l}} \delta Y^A_l\right]_{,k}. \tag{31}$$

If we introduce the expressions (25) and (26) for δY^A and δY^A_l into (31), we get an expression containing the ξ^i and their derivatives of the first and second order. After some rearrangements, and using the identity (30), the equation (27) may then be written in the form

$$-S_{i\ ,k}^{\ k} \xi^i - \left[S_i^{\ k} - V_{i\ ,l}^{\ kl}\right] \xi^i_{,k} + \left[V_i^{\ kl} + V_{i\ ,m}^{\ klm}\right] \xi^i_{,k,l} + V_i^{\ klm} \xi^i_{,k,l,m} = 0, \tag{32}$$

where we have used the abbreviations

$$S_i^{\ k} = -\frac{\delta V}{\delta Y^A} u^{Ak}_{\ \ i} + \left[\frac{\partial V}{\partial Y^A_k} - \left(\frac{\partial V}{\partial Y^A_{k,l}}\right)_{,l}\right] Y^A_i + \frac{\partial V}{\partial Y^A_{k,l}} Y^A_{i,l} - V \delta^k_i, \tag{33}$$

$$V_i^{\ kl} = \left[\frac{\partial V}{\partial Y^A_l} - \left(\frac{\partial V}{\partial Y^A_{l,m}}\right)_{,m}\right] u^{Ak}_{\ \ i} + \frac{\partial V}{\partial Y^A_{l,m}} \left(u^{Ak}_{\ \ i,m} - Y^A_i \delta^k_m\right), \tag{34}$$

$$V_i^{\ klm} = V_i^{\ kml} = u^{Ak}_{\ \ i} \frac{\partial V}{\partial Y^A_{l,m}}. \tag{35}$$

Since (32) has to hold for arbitrary choice of the functions $\xi^i(x)$, we get the following identities:

$$S_{i\ ,k}^{\ k} = 0, \tag{36}$$

$$S_i^{\ k} = V_{i\ ,l}^{\ kl}, \tag{37}$$

$$V_i^{\ kl} + V_i^{\ lk} + \left(V_i^{\ klm} + V_i^{\ lkm}\right)_{,m} = 0, \tag{38}$$

$$V_i^{\ klm} + V_i^{\ lmk} + V_i^{\ mkl} = 0, \quad V_i^{\ klm} = V_i^{\ kml}. \tag{39}$$

(36) shows that the quantity (33) satisfies a divergence relation and the method leads, apart from an arbitrary constant factor, uniquely to the expression (33). If we were only interested in deriving (33) and (36), we could have obtained this result much more easily by considering a "rigid infinitesimal parallel displacement" of the system of coordinates where the ξ^i are constants ε^i. In that case, we have by (25) and (26)

$$\delta Y^A = - Y^A_i \varepsilon^i, \quad \delta Y^A_l = - Y^A_{i,\,l} \varepsilon^i. \tag{40}$$

Introduction of these expressions into (31) and (27) gives directly, by means of the identity (30),

$$- \varepsilon^i S_{i,\,k}^{\,k} = 0, \tag{41}$$

which then leads to (36) on account of the arbitrariness in the choice of the constants ε^i.

From (39) we get

$$\left(V_i^{\,mkl} + \frac{1}{2} V_i^{\,klm}\right)_{,\,l,\,m} = \frac{1}{2}\left(V_i^{\,mkl} + V_i^{\,lkm} + V_i^{\,klm}\right)_{,\,l,\,m} = 0. \tag{42}$$

Thus, if we define a new quantity $U_i^{\,kl}$ by

$$U_i^{\,kl} = V_i^{\,kl} - \frac{2}{3}\left(V_i^{\,mkl} + \frac{1}{2} V_i^{\,klm}\right)_{,\,m} \tag{43}$$

we get by (37) and (42)

$$S_i^{\,k} = U_{i\;\,,\,l}^{\,kl}. \tag{44}$$

This expression has the advantage that $U_i^{\,kl}$ is antisymmetric in k, l so that (36) is an immediate consequence of (44).

In fact we have, by (43), (38) and (39),

$$U_i^{\,kl} + U_i^{\,lk} = V_i^{\,kl} + V_i^{\,lk} - \frac{2}{3}\left[V_i^{\,mkl} + V_i^{\,mlk} + \frac{1}{2}\left(V_i^{\,klm} + V_i^{\,lkm}\right)\right]_{,\,m}$$

$$= \left[V_i^{\,mkl} - \frac{4}{3} V_i^{\,mkl} - \frac{1}{3}\left(V_i^{\,klm} + V_i^{\,lmk}\right)\right]_{,\,m} = 0,$$

i. e.

$$U_i^{\,kl} = - U_i^{\,lk}. \tag{45}$$

We can therefore also write $U_i^{\ kl}$ in the manifestly antisymmetric form

$$\left.\begin{aligned} U_i^{\ kl} &= \frac{1}{2}(U_i^{\ kl} - U_i^{\ lk}) = \frac{1}{2}(V_i^{\ kl} - V_i^{\ lk}) - \frac{1}{6}(V_i^{\ klm} - V_i^{\ lkm})_{,\,m} \\ U_i^{\ kl} &= \frac{1}{2}\left\{\left(\left[\frac{\partial V}{\partial Y^A_l} - \left(\frac{\partial V}{\partial Y^A_{l,\,m}}\right)_{,\,m}\right] u^{Ak}{}_i + \frac{\partial V}{\partial Y^A_{l,\,m}} u^{Ak}{}_{i,\,m} - \frac{1}{3}\left[u^{Ak}{}_i \frac{\partial V}{\partial Y^A_{l,\,m}}\right]_{,\,m}\right) - \left(\overset{\leftarrow}{\underset{\rightarrow}{k,\,l}}\right)\right\} \end{aligned}\right\} \quad (46)$$

where the last term is obtained from the first by interchanging the indices k and l.

Thus, the method of infinitesimal transformations leads (apart from an arbitrary constant factor) uniquely to a quantity $S_i^{\ k}$ which satisfies the divergence relation (36) and which, by (44), is derivable from a "super-potential" defined by (43), (34), (35) or (46).

The preceding considerations are easily generalized to the case where V is a function of the Y^A and their derivatives of arbitrarily high order. The variational derivative of V with respect to Y^A is here defined as

$$\frac{\delta V}{\delta Y^A} = \sum_{n=0} (-1)^n \left(\frac{\partial V}{\partial Y^A_{i_1,\,i_2,\,\dots\,i_n}}\right)_{,\,i_1,\,i_2,\,\dots\,i_n}. \quad (47)$$

(Summation over n and for each n independent summation over the indices i_1, i_2, ... i_n!)

(47) obviously reduces to (18), if V does not depend on derivatives of Y^A of higher than the second order. Similarly, we introduce the variational derivatives of V with respect to Y^A_i, $Y^A_{i,\,k}$, ... by

$$\left.\begin{aligned} \frac{\delta V}{\delta Y^A_i} &= \frac{\partial V}{\partial Y^A_i} - \left(\frac{\partial V}{\partial Y^A_{i,\,i_1}}\right)_{,\,i_1} + \cdots = \sum_{n=0} (-1)^n \left(\frac{\partial V}{\partial Y^A_{i,\,i_1,\,\dots\,i_n}}\right)_{,\,i_1,\,\dots\,i_n} \\ \frac{\delta V}{\delta Y^A_{i,\,k}} &= \frac{\partial V}{\partial Y^A_{i,\,k}} - \left(\frac{\partial V}{\partial Y^A_{i,\,k,\,i_1}}\right)_{,\,i_1} + \cdots = \sum_{n=0} (-1)^n \left(\frac{\partial V}{\partial Y^A_{i,\,k,\,i_1,\,\dots\,i_n}}\right)_{,\,i_1,\,\dots\,i_n} \end{aligned}\right\} \quad (48)$$

etc. In this general case, the method of infinitesimal transformations leads to the following energy-momentum complex $S_i^{\ k}$ satisfying the divergence relation (36):

$$\left.\begin{aligned} S_i^{\ k} &= -\frac{\delta V}{\delta Y^A} u^{Ak}_{\ \ i} + \frac{\delta V}{\delta Y^A_k} Y^A_i + \frac{\delta V}{\delta Y^A_{k, l_1}} Y^A_{i. l_1} + \cdots - \delta^k_i V \\ &= -\frac{\delta V}{\delta Y^A} u^{Ak}_{\ \ i} + \sum_{n=0} \left(\frac{\delta V}{\delta Y^A_{k, l_1, \ldots l_n}}\right) Y^A_{i, l_1, \ldots l_n} - \delta^k_i V. \end{aligned}\right\} \tag{49}$$

In the first place, it is clear that the identity (30) still is true, since the considerations in connection with the equations (27 – (30) are valid also here. Then, if we consider a rigid displacement with $\xi^i = \varepsilon^i =$ constant, where

$$\delta Y^A_{l_1, \ldots l_n} = - Y^A_{i, l, \ldots l_n} \varepsilon^i, \tag{50}$$

one easily finds that the equations (41), and consequently (36), hold also in this case with $S_i^{\ k}$ given by (49).

3. Gravitational Fields

It is well known that the gravitational field may be treated as a Lagrangean system with the Lagrangean density

$$\mathfrak{L} = \sqrt{-g}\, g^{ik} (\Gamma^l_{ik} \Gamma^m_{lm} - \Gamma^m_{il} \Gamma^l_{km}), \tag{51}$$

the Γ^i_{kl} being the Christoffel symbols. In fact we have for all variations of the field variables g^{ik} which vanish at the surface of a region Ω in 4-space

$$\delta \int_\Omega \mathfrak{L}\, dx = \int_\Omega \frac{\delta \mathfrak{L}}{\delta g^{ik}} \delta g^{ik}\, dx = \int_\Omega \mathfrak{G}_{ik} \delta g^{ik}\, dx, \tag{52}$$

where

$$\mathfrak{G}_{ik} = \sqrt{-g}\, G_{ik} = \sqrt{-g}\, (R_{ik} - \frac{1}{2} g_{ik} R). \tag{53}$$

Therefore, the field equations are of the form

$$\frac{1}{\varkappa} \frac{\delta \mathfrak{L}}{\delta g^{ik}} = \frac{1}{\varkappa} \mathfrak{G}_{ik} = -\sqrt{-g}\, T_{ik}. \tag{54}$$

Comparing (54) with (17) we see that we are dealing with a special case of the systems treated in section 2. The field variables Y^A are here the quantities g^{ik}, and $V = \mathfrak{L}/\varkappa$ is a function of the Y^A and their first derivatives only. Hence,

$$\frac{\delta V}{\delta Y^A} = \frac{1}{\varkappa}\frac{\delta \mathfrak{L}}{\delta g^{ik}} = \frac{1}{\varkappa}\left[\frac{\partial \mathfrak{L}}{\partial g^{ik}} - \left(\frac{\partial \mathfrak{L}}{\partial g^{ik}_l}\right)_{,l}\right] \tag{55}$$

Since $g^{ik} = g^{ki}$, we have here a case where some of the Y^A are equal. Thus, with the convention mentioned on page 7 equations (22) hold, i. e.,

$$\frac{\partial \mathfrak{L}}{\partial g^{ik}} = \frac{\partial \mathfrak{L}}{\partial g^{ki}}, \quad \frac{\delta \mathfrak{L}}{\delta g^{ik}} = \frac{\delta \mathfrak{L}}{\delta g^{ik}}. \tag{56}$$

However, $V = \mathfrak{L}/\varkappa$ is a scalar density only under *linear* space-time transformations. Therefore, only the identities (36) and (37) can be derived in this case, since $\xi^i_{,k,l} = \xi^i_{,k,l,m} = 0$ for linear transformation, which means that the last two terms in (32) are missing. A simple calculation shows that $S_i^{\ k}$ and $V_i^{\ kl}$ in this case are

$$S_i^{\ k} = 2\,\Theta_i^{\ k}, \quad V_i^{\ kl} = 2\,s_i^{\ kl} \tag{57}$$

with $\Theta_i^{\ k}$ and $s_i^{\ kl}$ given by (1), (3), and (4). Furthermore, (30) becomes identical with the contracted Bianchi identities

$$G^k_{i;\,k} \equiv \frac{1}{\sqrt{-g}}\,\frac{\partial \sqrt{-g}\,G^k_i}{\partial x^k} + \frac{1}{2}\,g^{rs}_i\,G^{rs} = 0. \tag{58}$$

However, this equation cannot here be derived by the method used in section 2, since this would require invariance of $\int_\Omega V dx = \frac{1}{\varkappa}\int_\Omega \mathfrak{L}\,dx$ under *arbitrary* space-time transformations.

We get a more satisfactory description by treating the gravitational field as a non-Lagrangean system of the type considered in section **2** with

$$V = \mathfrak{R}/\varkappa = \sqrt{-g}\,R/\varkappa \tag{59}$$

which is a function of the g^{ik} and their first *and* second order derivatives g^{ik}_l and $g^{ik}_{l,m}$. Also in this case we have an equation of the type (52), i. e.,

$$\delta \int_{\Omega} \mathfrak{R}\, dx = \int_{\Omega} \frac{\delta \mathfrak{R}}{\delta g^{ik}} \delta g^{ik}\, dx = \int_{\Omega} \mathfrak{G}_{ik}\, \delta g^{ik}\, dx \tag{60}$$

i. e.,

$$\frac{\delta \mathfrak{R}}{\delta g^{ik}} = \mathfrak{G}_{ik}. \tag{61}$$

The reason for this is that $\mathfrak{R}$ differs from $\mathfrak{L}$ by a divergence term only. In fact, we have

$$\left.\begin{aligned} \mathfrak{R} &= \mathfrak{L} + \mathfrak{h}, \\ \mathfrak{h} &= \varkappa h_{r,}{}^{rl}{}_{l}. \end{aligned}\right\} \tag{62}$$

This follows at once from (1) and (8) if we remark that

$$\left.\begin{aligned} \Theta_r{}^r = \sqrt{-g}\,(T_r^r + \vartheta_r{}^r) &= \frac{\sqrt{-g}}{\varkappa}\, G_r^r + \frac{1}{2\varkappa}\left[\frac{\partial \mathfrak{L}}{\partial g_r^{lm}}\, g_r^{lm} - 4\,\mathfrak{L}\right] \\ &= \frac{1}{\varkappa}(\mathfrak{R} - \mathfrak{L}). \end{aligned}\right\} \tag{63}$$

In the last equation, we have used the fact that $\mathfrak{L}$ is a homogeneous function of the g_r^{lm} of degree 2. With $h_i{}^{kl}$ given by (7) a simple calculation shows that

$$h_r{}^{rl} = \frac{1}{\varkappa}\left(\sqrt{-g}\, g_m^{lm} + 2\, g^{lm}(\sqrt{-g})_{,\,m}\right) = \frac{1}{\varkappa\sqrt{-g}}\left(-g\, g^{lm}\right)_{,\,m} \tag{64}$$

(see, for instance, the Appendix of reference [4]).

For arbitrary variations δg^{ik} which vanish at the surface of Ω, we now have

$$\delta \int_{\Omega} \mathfrak{h}\, dx = \int_{\Omega} \frac{\delta \mathfrak{h}}{\delta g^{ik}} \delta g^{ik}\, dx = \varkappa \int_{\Omega} (\delta\, h_r{}^{rl})_{,\,l}\, dx = 0, \tag{65}$$

i. e.,

$$\frac{\delta \mathfrak{h}}{\delta g^{ik}} = 0. \tag{66}$$

With

$$V = \frac{1}{\varkappa} \mathfrak{R} \tag{67}$$

the field equations take the form (17), i. e.,

$$\frac{1}{\varkappa}\mathfrak{G}_{ik} = \frac{1}{\varkappa}\frac{\delta\mathfrak{L}}{\delta g^{ik}} = \frac{1}{\varkappa}\frac{\delta\mathfrak{R}}{\delta g^{ik}} = -\sqrt{-g}\,T_{ik}\,. \tag{68}$$

Since $\mathfrak{R}$ is a scalar density under arbitrary space-time transformations, all the relations (32)–(46) of section **2** are valid here. For an infinitesimal transformation (23), we now have

$$\delta g^{rs} = g^{sk}\,\xi^r_{,\,k} + g^{rk}\,\xi^s_{,\,k} - g^{rs}_i\,\xi^i \tag{69}$$

which, by comparison with (25), gives

$$u^{Ak}{}_i = u^{rsk}{}_i = \delta^r_i\,g^{sk} + g^{rk}\,\delta^s_i. \tag{70}$$

Then we get in the first place from (30), (68), and (70)

$$\frac{2}{\varkappa}\left[\frac{\partial\,\mathfrak{G}^k_i}{\partial x^k} + \frac{1}{2}g^{rs}_i\,\mathfrak{G}_{rs}\right] = 0\,, \tag{71}$$

i. e., the Bianchi identity (58). Next, by (33), (59), (61), (62), and (70)

$$\left.\begin{aligned} S_i{}^k = -\frac{2}{\varkappa}\mathfrak{G}^k_i + \frac{1}{\varkappa}\left[\frac{\partial\,\mathfrak{L}}{\partial g^{lm}_k}\,g^{lm}_i - \delta^k_i\,\mathfrak{L}\right] + \frac{1}{\varkappa}\left[\frac{\partial\,\mathfrak{h}}{\partial g^{rs}_k} - \left(\frac{\partial\,\mathfrak{h}}{\partial g^{rs}_{k,\,l}}\right)_{,\,l}\right]g^{rs}_i \\ + \frac{1}{\varkappa}\left(\frac{\partial\,\mathfrak{h}}{\partial g^{rs}_{k,\,l}}\right)g^{rs}_{i,\,l} - \frac{1}{\varkappa}\,\delta^k_i\,\mathfrak{h} \end{aligned}\right\} \tag{72}$$

or, using (3) and the field equations (2),

$$S_i{}^k = \sqrt{-g}\left(T^k_i + 2\,\vartheta_i{}^k - \frac{1}{\varkappa}\,G^k_i\right) + A^k_i - K^{kl}_{i,\,l} \tag{73}$$

with

$$\varkappa A^k_i = \left[\frac{\partial\,\mathfrak{h}}{\partial g^{rs}_k} - \left(\frac{\partial\,\mathfrak{h}}{\partial g^{rs}_{k,\,l}}\right)_{,\,l}\right]g^{rs}_i - \left(\frac{\partial\,\mathfrak{h}}{\partial g^{rs}_{k,\,l}}\right)_{,\,i}g^{rs}_l + \left[\left(\frac{\partial\,\mathfrak{h}}{\partial g^{rs}_{l,\,m}}\,g^{rs}_l\right)_{,\,m} - \mathfrak{h}\right]\delta^k_i \tag{74}$$

$$K^{kl}_i = \frac{1}{\varkappa}\left(\delta^k_i\,\frac{\partial\,\mathfrak{h}}{\partial g^{rs}_{l,\,m}} - \delta^l_i\,\frac{\partial\,\mathfrak{h}}{\partial g^{rs}_{k,\,m}}\right)g^{rs}_m = -K^{lk}_i. \tag{75}$$

As shown in Appendix A, A^k_i is identically zero with $\mathfrak{h}$ given by (62) and (64), and K^{kl}_i becomes

$$K^{kl}_i = \delta^k_i\,h_r{}^{rl} - \delta^l_i\,h_r{}^{rk}. \tag{76}$$

Hence,

$$S_i^{\ k} = \sqrt{-g}\,(T_i^k + t_i^{\ k}), \tag{77}$$

where

$$\sqrt{-g}\,t_i^{\ k} = -\frac{1}{\varkappa}\mathfrak{G}_i^k + 2\sqrt{-g}\,\vartheta_i^{\ k} - (\delta_i^k h_r^{\ rl} - \delta_i^l h_r^{\ rk})_{,\,l}, \tag{78}$$

i. e., the quantity defined by (11). Thus, the "conserved" quantity $S_i^{\ k}$ is in this case just the pseudo-tensor density of energy and momentum defined by (10).

For the superpotential $U_i^{\ kl}$ we get by (46), (62), and (70)

$$\varkappa U_i^{\ kl} = \left(\frac{\partial \mathfrak{L}}{\partial g_l^{im}} g^{km} + \left[\frac{\partial \mathfrak{h}}{\partial g_l^{is}} - \left(\frac{\partial \mathfrak{h}}{\partial g_{l,\,m}^{is}}\right)_{,\,m}\right] g^{ks} + \frac{\partial \mathfrak{h}}{\partial g_{l,\,m}^{is}} g_m^{ks} - \frac{1}{3}\left[\frac{\partial \mathfrak{h}}{\partial g_{l,\,m}^{is}} g^{ks}\right]_{,\,m}\right) - \left(\overleftrightarrow{k,\,l}\right), \tag{79}$$

$$U_i^{\ kl} = s_i^{\ kl} - s_i^{\ lk} + B_i^{kl} - B_i^{lk}, \tag{80}$$

where $s_i^{\ kl}$ is the quantity given by (4) – (7) and

$$\varkappa B_i^{kl} = \left[\frac{\partial \mathfrak{h}}{\partial g_l^{is}} - \left(\frac{\partial \mathfrak{h}}{\partial g_{l,\,m}^{is}}\right)_{,\,m}\right] g^{ks} + \frac{\partial \mathfrak{h}}{\partial g_{l,\,m}^{is}} g_m^{ks} - \frac{1}{3}\left(\frac{\partial \mathfrak{h}}{\partial g_{l,\,m}^{is}} g^{ks}\right)_{,\,m}. \tag{81}$$

The calculation of B_i^{kl} and of $U_i^{\ kl}$ is completed in Appendix A and the result is that $U_i^{\ kl}$ in the present case is equal to the superpotential $\chi_i^{\ kl}$ given by (13), which makes the equation (44) identical with the equation (12) for $\mathcal{T}_i^{\ k}$. Thus, the method of infinitesimal transformations leads, apart from an arbitrary constant factor, directly to the expressions (10) – (13) for the pseudo-tensor density of energy and momentum. The arbitrary factor is fixed by the condition that the integrals P_i in (14) for "closed" systems must have the right values and it turns out that, with $V = \mathfrak{R}/\kappa$, this factor has to be equal to one.

4. The Matter Field

We shall now assume that the "matter" which produces the gravitational field has the character of a tensor field described by a number of field variables $Q^a(x)$. (For simplicity, we exclude spinors). Further, we assume that this field is of the Lagrangean type, i. e., the matter field equations are of the form

$$\frac{\delta \mathfrak{M}}{\delta Q^a} = 0, \tag{82}$$

where $\mathfrak{M}$ is a scalar density depending on the $Q^a(x)$, $g^{ik}(x)$ and their first order derivatives, and the symmetrical matter tensor is obtained by derivation of $\mathfrak{M}$ with respect to g^{ik}, i. e.,

$$\frac{\delta \mathfrak{M}}{\delta g^{ik}} = \sqrt{-g}\, T_{ik}. \tag{83}$$

In all practical cases, $\mathfrak{M}$ does even not depend on g^{ik}_l:

$$\mathfrak{M} = \mathfrak{M}(g^{ik},\ Q^a,\ Q^a_{,i}). \tag{84}$$

This implies that both the gravitational field equations (2) and the matter field equations (82) are derivable from the variational principle

$$\delta \int \left(\frac{1}{\varkappa} \mathfrak{R} + \mathfrak{M} \right) dx = 0 \tag{85}$$

for independent variations of the g^{ik} and the Q^a.

We may now apply the general considerations of section **2** to the non-closed system with

$$\left. \begin{aligned} V &= \mathfrak{M} \\ Y^A &= \left\{ g^{ik},\ Q^a \right\}. \end{aligned} \right\} \tag{86}$$

As we shall see now, this leads to a special case of the well-known connection between the symmetrical and the "canonical" matter tensor discovered by Rosenfeld and by Belinfante [6]. With $V = \mathfrak{M}(g^{ik}, Q^a, Q^a_{,i})$ we get from (33), (70), and (83)

$$S_i^{\ k} = -2\sqrt{-g}\, T_i^k - \frac{\delta \mathfrak{M}}{\delta Q^a} u^{ak}_{\ \ i} + \frac{\partial \mathfrak{M}}{\partial Q^a_k} Q^a_{,i} - \mathfrak{M}\, \delta_i^k. \tag{87}$$

On the other hand, we have, by (44) and (46), in the present case

$$\left. \begin{aligned} U_i^{\ kl} &= \frac{1}{2} \left[\frac{\partial \mathfrak{M}}{\partial Q^a_{,l}} u^{ak}_{\ \ i} - \frac{\partial \mathfrak{M}}{\partial Q^a_{,k}} u^{al}_{\ \ i} \right], \\ S_i^{\ k} &= U_i^{\ kl}{}_{,l}. \end{aligned} \right\} \tag{88}$$

Thus, we get from (87) and (88)

$$\sqrt{-g}\, T_i^k = \sqrt{-g}\,(\breve{T}_i^k + \widehat{T}_i^k) \tag{89}$$

with

$$\sqrt{-g}\, \breve{T}_i^k = \frac{\partial\, \mathfrak{M}/2}{\partial\, Q^a_{,\,k}}\, Q^a_{,\,i} - (\mathfrak{M}/2)\, \delta_i^k \tag{90}$$

$$\sqrt{-g}\, \widehat{T}_i^k = -\frac{\delta\, \mathfrak{M}/2}{\delta\, Q^a}\, u^{ak}{}_i - \frac{1}{2}\left[\frac{\partial\, \mathfrak{M}/2}{\partial\, Q^a_{,\,l}}\, u^{ak}{}_i - \frac{\partial\, \mathfrak{M}/2}{\partial\, Q^a_{,\,k}}\, u^{al}{}_i\right]_{,\,l} \tag{91}$$

$\breve{T}_i{}^k$ is the generally unsymmetric canonical matter tensor derivable from the "matter Lagrangean"

$$\mathfrak{L}^{(m)} = -\,\mathfrak{M}/2 \tag{92}$$

and $\widehat{T}_i{}^k$ is the term which has to be added to $\breve{T}_i{}^k$ in order to give the symmetrical matter tensor T_i^k. The first term in (91) is zero on account of the field equations (82) and, for a matter-system confined to a final part of 3-space, the last term will give no contribution to the total matter energy and momentum. In fact, we have

$$\left.\begin{aligned} P_i^{(m)} &= \frac{1}{c}\int \sqrt{-g}\, T_i^4\, dx^1\, dx^2\, dx^3 = \frac{1}{c}\int \sqrt{-g}\, \breve{T}_i{}^4\, dx^1\, dx^2\, dx^3 \\ &\quad - \frac{1}{2\,c}\int U_i{}^{4\lambda}{}_{,\,\lambda}\, dx^1\, dx^2\, dx^3 = \frac{1}{c}\int \sqrt{-g}\, \breve{T}_i{}^4\, dx^1\, dx^2\, dx^3. \end{aligned}\right\} \tag{93}$$

In general, $P_i^{(m)}$ is not constant in time. Only the sum of the matter part and the gravitational part, i. e.,

$$P_i = \frac{1}{c}\int T_i^4\, dx^1\, dx^2\, dx^3 = \frac{1}{c}\int \sqrt{-g}\,\left[T_i^4 + t_i{}^4\right] dx^1\, dx^2\, dx^3 \tag{94}$$

is conserved for a closed system.

As an example, we consider the case where the matter field is a purely electromagnetic field. Here, we have

$$\left.\begin{aligned} \mathfrak{M} &= \frac{1}{2}\sqrt{-g}\, g^{rl}\, g^{sm}\, F_{rs} F_{lm} \\ F_{ik} &= A_{k,\,i} - A_{i,\,k} = -F_{ki}. \end{aligned}\right\} \tag{95}$$

As the field variables Q^a we may take the components A_i of the four-potential. Then, we get

$$\frac{\partial \mathfrak{M}}{\partial A_i} = 0, \qquad \frac{\partial \mathfrak{M}}{\partial A_{i,k}} = -2\sqrt{-g}\, F^{ik} \tag{96}$$

and the field equations (82) are the Maxwell equations

$$\frac{\delta \mathfrak{M}}{\delta A_i} = -\frac{\partial}{\partial x^k}\left(2\sqrt{-g}\, F^{ik}\right) = 0. \tag{97}$$

Further, since for any variation of the g^{ik}

$$\delta\left(\sqrt{-g}\, g^{rl} g^{sm}\right) = \frac{\sqrt{-g}}{2}\left[-g_{ik}\, g^{rl} g^{sm} + g^{sm}(\delta_i^r \delta_k^l + \delta_k^r \delta_i^l) + g^{rl}(\delta_i^s \delta_k^m + \delta_k^s \delta_i^m)\right] \delta g^{ik},$$

we have, by (83)

$$\left.\begin{aligned}\sqrt{-g}\, T_{ik} &= \frac{\partial \mathfrak{M}}{\partial g^{ik}} = \frac{1}{2} F_{rs} F_{lm} \frac{\partial\left(\sqrt{-g}\, g^{rl} g^{sm}\right)}{\partial g^{ik}} \\ &= \sqrt{-g}\left[F_{il} F_k^{\ l} - \frac{1}{4} g_{ik} F_{lm} F^{lm}\right]\end{aligned}\right\} \tag{98}$$

which is the usual expression for the electromagnetic energy-momentum tensor.

On the other hand, the canonical tensor is, by (90), (95), and (96),

$$\sqrt{-g}\, \breve{T}_i^{\ k} = \frac{\partial \mathfrak{M}/2}{\partial A_{l,k}} A_{l,i} - (\mathfrak{M}/2)\, \delta_i^k = -\sqrt{-g}\, F^{lk} A_{l,i} - \frac{\sqrt{-g}}{4} F_{lm} F^{lm} \delta_i^k. \tag{99}$$

It differs from (98) by the term $\sqrt{-g}\, \widehat{T}_i^{\ k}$ given by (91). Since A_i is a four-vector, we have for an infinitesimal transformation

$$\delta A_r = -A_i \xi^i_{\ ,r} - A_{r,i} \xi^i = u_{r\ i}^{\ k} \xi^i_{\ ,k} - A_{r,i} \xi^i, \tag{100}$$

i. e.,

$$u^{ak}_{\ \ i} \equiv u_{r\ i}^{\ k} = -\delta_r^k A_i.$$

Thus, by (91) and (96),

$$\sqrt{-g}\,\widehat{T}_i{}^k = -\left(\frac{\partial\,\mathfrak{M}/2}{\partial A_{r,l}}\right)_{,l}\delta_r^k A_i + \frac{1}{2}\left[\frac{\partial\,\mathfrak{M}/2}{\partial A_{r,l}}\,\delta_r^k A_i - \frac{\partial\,\mathfrak{M}/2}{\partial A_{r,k}}\,\delta_r^l A_i\right]_{,l}$$

$$= \left(\sqrt{-g}\,F^{kl}\right)_{,l} A_i - \frac{1}{2}\left[\sqrt{-g}\,F^{kl} A_i - \sqrt{-g}\,F^{lk} A_i\right]_{,l}$$

or

$$\sqrt{-g}\,\widehat{T}_i{}^k = -\sqrt{-g}\,F^{kl} A_{i,l}. \tag{101}$$

By adding the expressions (99) and (101) we get again the expression (98) for the symmetrical matter tensor, in accordance with the general equation (89).

In conclusion, we summarize the main results of the preceding sections. The total pseudo-tensor density of energy and momentum $\mathcal{T}_i{}^k$ may be written as the sum of a "matter part" T_i^k and a "gravitational part" $t_i{}^k$:

$$\mathcal{T}_i{}^k = \sqrt{-g}\,[T_i^k + t_i{}^k] \tag{102}$$

where, by (89) – (92)

$$\sqrt{-g}\,T_i{}^k = -Q^a_{,i}\frac{\partial\,\mathfrak{L}^{(m)}}{\partial\,Q^a_{,k}} + \delta_i^k\,\mathfrak{L}^{(m)} + \frac{1}{2}\left[u^{ak}_{\;\;i}\frac{\partial\,\mathfrak{L}^{(m)}}{\partial\,Q^a_{,l}} - u^{al}_{\;\;i}\frac{\partial\,\mathfrak{L}^{(m)}}{\partial\,Q^a_{,k}}\right]_{,l} + u^{ak}_{\;\;i}\frac{\delta\,\mathfrak{L}^{(m)}}{\delta\,Q^a} \tag{103}$$

and, by (72) – (78),

$$\left.\begin{aligned}\sqrt{-g}\,t_i{}^k &= -\frac{1}{\varkappa}\,\mathfrak{G}_i^k + 2\sqrt{-g}\,\vartheta_i{}^k - \left(\delta_i^k\,h_r{}^{rl} - \delta_i^l\,h_r{}^{rk}\right)_{,l}\\ &= \frac{1}{\varkappa}\left\{-\frac{\delta\,\mathfrak{R}}{\delta\,g^{il}}\,g^{kl} + \left[\frac{\partial\,\mathfrak{R}}{\partial\,g^{rs}_k} - \left(\frac{\partial\,\mathfrak{R}}{\partial\,g^{rs}_{k,l}}\right)_{,l}\right]g^{rs}_i + \frac{\partial\,\mathfrak{R}}{\partial\,g^{rs}_{k,l}}\,g^{rs}_{i,l} - \mathfrak{R}\,\delta_i^k\right\}.\end{aligned}\right\} \tag{104}$$

Apart from the last two terms which give no contribution to $P_i^{(m)}$, the matter part (103) has the canonical form corresponding to a matter Lagrangean $\mathfrak{L}^{(m)} = -\mathfrak{M}/2$. On the other hand, the gravitational part (104) has an entirely different structure and it can *not* be derived from a Lagrangean density according to the usual rules. This may be taken as an indication that the "quantization" of gravitational fields should be performed in a way which differs from the usual rules of ordinary quantum mechanics. It is true that $\sqrt{-g}\,t_i{}^k$ in (104) may also be written

$$\sqrt{-g}\,t_i{}^k = \sqrt{-g}\,\vartheta_i{}^k + \psi_i{}^{kl}{}_{,l}, \tag{105}$$

where

$$\psi_i^{\ kl} = h_i^{\ kl} - \delta_i^k h_r^{\ rl} + \delta_i^l h_r^{\ rk} \tag{106}$$

is antisymmetric in k and l. Therefore, by partial integrations, the total gravitational momentum and energy take the form

$$P_i^{(g)} = \frac{1}{c}\int \sqrt{-g}\, t_i^{\ 4}\, dx^1\, dx^2\, dx^3 = \frac{1}{c}\int \sqrt{-g}\, \vartheta_i^{\ k}\, dx^1\, dx^2\, dx^3 + A_i, \tag{107}$$

where A_i depends on the gravitational variables at spatial infinity only. Formally the gravitational field may therefore be treated as a canonical system. But quite apart from the complications, already present on the "classical" level, which occur in the transition from the Lagrangean to the Hamiltonean form due to the different types of restraints [(7)], we are faced with the difficult problem of finding the correct order of factors in the transition to a quantal description along the usual lines of quantum mechanics. Also it should be mentioned that the division of $\mathcal{T}_i^{\ k}$ into a matter part and a gravitational part is to a large extent arbitrary due to the fact that the matter tensor is the source of the gravitational field. By means of these equations, a larger or smaller part of T_i^k may be eliminated in $\mathcal{T}_i^{\ k}$. If we eliminate T_i^k entirely, we arrive at the simple and convenient expression (12) which depends on the gravitational field variables only.

For the total momentum and energy of the system, we then get

$$P_i = \frac{1}{c}\int \mathcal{T}_i^{\ 4}\, dx^1\, dx^2\, dx^3 = \frac{1}{c}\int \chi_i^{\ 4\lambda}{}_{,\lambda}\, dx^1\, dx^2\, dx^3$$

which, by means of Gauss' theorem, may be written as a surface integral depending only on the gravitational field variables at spatial infinity.

5. Transformation Properties of $\mathcal{T}_i^{\ k}$ and $t_i^{\ k}$

The energy-momentum complex is a tensor density under *linear* transformations, only. We shall now investigate the transformation properties of $\mathcal{T}_i^{\ k}$ and $t_i^{\ k}$ under the most general space-time transformations. To this end, we consider an arbitrary vector field $a^i(x)$ and the antisymmetrical tensor density

$$\left.\begin{aligned} \mathfrak{F}^{kl} &= -\mathfrak{F}^{lk} = \frac{\sqrt{-g}}{\varkappa}\,(a_{n,\,m} - a_{m,\,n})\, g^{km} g^{ln}, \\ a_i(x) &= g_{ik}\, a^k(x). \end{aligned}\right\} \tag{108}$$

Further, the vector density

$$\mathfrak{F}^k = \mathfrak{F}^{kl}{}_{,\,l} \tag{109}$$

which, on account of the antisymmetry of $\mathfrak{F}^{kl}$ in k and l, satisfies the divergence relation

$$\mathfrak{F}^k_{,\,k} = 0 \tag{110}$$

for arbitrary vector fields $a^i(x)$.

As remarked by Komar(8), the vector density $\mathfrak{F}^k$ is closely related to the complex $\mathcal{T}_i{}^k$. In fact, if we let the contravariant components of the arbitrary vector field a^i be constants ε^i in a definite system of coordinates, we have by (108), (109), (12), and (13)

$$\mathfrak{F}^k = \varepsilon^i \left[\frac{\sqrt{-g}}{\varkappa}\,(g_{in,\,m} - g_{im,\,n})\, g^{km} g^{ln}\right]_{,\,l} = \varepsilon^i \chi_i{}^{kl}{}_{,\,l} = \varepsilon^i \mathcal{T}^k_i. \tag{111}$$

For an arbitrary vector field $a^i(x)$, we get on the other hand, since

$$\left.\begin{aligned} a_{n,\,m} &= a^i g_{in,\,m} + a^i_{,\,m} g_{in}, \\ \mathfrak{F}^{kl} &= a^i \chi_i{}^{kl} - a^i_{,\,m}\, b_i{}^{klm} \end{aligned}\right\} \tag{112}$$

where $b_i{}^{klm}$ is the tensor density of rank 4 defined by

$$b_i{}^{klm} = -b_i{}^{lkm} = \frac{\sqrt{-g}}{K}\,(\delta^k_i g^{lm} - \delta^l_i g^{km}). \tag{113}$$

Hence, by (109), (112), and (12),

$$\mathfrak{F}^k = a^i \mathcal{T}_i{}^k + a^i_{,\,l}\,(\chi^{kl}_i - b_i{}^{kml}{}_{,\,m}) - a^i_{,\,l,\,m}\, b^{klm}_i. \tag{114}$$

Now, consider an arbitrary space-time transformation $(x^i) \to (x'^i)$ with

$$\alpha^i_k(x) = \frac{\partial x'^i}{\partial x^k}, \quad \breve{\alpha}^i_k(x') = \frac{\partial x^i}{\partial x'^k}. \tag{115}$$

Since $\mathfrak{F}^k$ is a vector density, we have

$$\mathfrak{F}'^k = |\breve{\alpha}|\,\alpha^k_l\,\mathfrak{F}^l = |\breve{\alpha}|\,\alpha^k_l\{a^r\,\mathcal{T}_r{}^l + a^r{}_{,m}(\chi_r{}^{lm} - b_r{}^{lnm}{}_{,n}) - a^r{}_{,m,n}\,b_r{}^{lmn}, \quad (116)$$

where $|\breve{\alpha}| = \det\{\breve{\alpha}^i_k\}$ is the determinant of the matrix $\breve{\alpha}^i_k$. If we choose the components a'^i in the primed system to be constants ε^i, we have

$$\left.\begin{aligned} \mathfrak{F}'^k &= \varepsilon^i\,\mathcal{T}'^k_i \\ a^r &= \breve{\alpha}^r_i\,\varepsilon^i \\ a^r{}_{,m} &= \alpha^s_m\,\breve{\alpha}^r_{i,s}\,\varepsilon^i \\ a^r{}_{,m,n} &= \alpha^s_{m,n}\,\breve{\alpha}^r_{i,s}\,\varepsilon^i + \alpha^s_m\,\alpha^t_n\,\breve{\alpha}^r_{i,s,t}\,\varepsilon^i, \end{aligned}\right\} \quad (117)$$

where

$$\breve{\alpha}^r_{i,s} = \frac{\partial^2 x^r}{\partial x'^i\,\partial x'^s}, \quad \alpha^s_{m,n} = \frac{\partial^2 x'^s}{\partial x^m\,\partial x^n}, \quad \breve{\alpha}^r_{i,s,t} = \frac{\partial^3 x^r}{\partial x'^i\,\partial x'^s\,\partial x'^t}, \ldots \quad (118)$$

Introduction of (117) into (116) gives, since the constants ε^i are arbitrary, the following transformation law for the complex $\mathcal{T}_i{}^k$:

$$\mathcal{T}'^k_i = |\breve{\alpha}|\,\alpha^k_l\,\breve{\alpha}^m_i\,\mathcal{T}_m{}^l + |\breve{\alpha}|\,\alpha^k_l\{\breve{\alpha}^r_{i,s}\alpha^s_m(\chi_r{}^{lm} - b_r{}^{lnm}{}_{,n}) - (\breve{\alpha}^r_{i,s,t}\alpha^s_m\,\alpha^t_n + \breve{\alpha}^r_{i,s}\alpha^s_{m,n})\,b_r{}^{lmn}\} \quad (119)$$

The last term on the right-hand side of (119) represents the deviation from the transformation law of a tensor density. For linear transformations, where $\breve{\alpha}^r_{i,s} = \breve{\alpha}^r_{i,s,t} = 0$, this term is zero, in accordance with the fact that $\mathcal{T}_i{}^k$ is an affine tensor density. Moreover, for the purely spatial transformation (15), we have

$$\breve{\alpha}^r_4 = \frac{\partial x^r}{\partial x'^4} = \delta^r_4, \quad \breve{\alpha}^r_{4,s} = 0, \quad (120)$$

i. e.,

$$\mathcal{T}'^k_4 = |\breve{\alpha}|\,\alpha^k_l\,\mathcal{T}_4{}^l. \quad (121)$$

This equation shows that the fourth column of the matrix $\mathcal{T}_i{}^k$ transforms like a vector density under the transformation (15), a property which was the starting point in our derivation of $\mathcal{T}_i{}^k$ in [(4)]. The apparent distinction of the time direction revealed in this property is not surprising, since the densities of energy and energy current in this description are connected with the "time column" of $\mathcal{T}_i{}^k$. Actually, if a is any fixed value of the indices 1, 2, 3, 4, the a^{th} column $\mathcal{T}_a^{\ k}$ will also transform like a vector density under the transformations

$$x'^a = x^a, \quad x'^i = f^i(x^k) \quad \text{with} \quad i \neq a, \quad k \neq a; \tag{122}$$

for, in that case, we have

$$\left.\begin{aligned} \breve{\alpha}_a^r &= \frac{\partial x^r}{\partial x'^a} = \delta_a^r \\ \mathcal{T}'_a{}^k &= \left|\breve{\alpha}\right| \alpha_l^k \, \mathcal{T}_a^{\ l}. \end{aligned}\right\} \tag{123}$$

But, for $a \neq 4$, this property does not lend itself to a simple physical interpretation.

Since the matter part $\sqrt{-g}\, T_i^k$ of $\mathcal{T}_i^{\ k}$ transforms like a tensor density, we get for the gravitational complex $t_i^{\ k}$ defined by (10) and (11) the transformation law

$$t_i'^k = \alpha_l^k \breve{\alpha}_i^m t_m^{\ l} + \alpha_l^k \left\{ \breve{\alpha}_{i,\,s}^r \alpha_m^s \left(\chi_r^{\ lm} - b_r^{\ lnm}{}_{,\,n}\right) - \left(\breve{\alpha}_{i,\,s,\,t}^r \alpha_m^s \alpha_n^t + \breve{\alpha}_{i,\,s}^r \alpha_{m,\,n}^s\right) b_r^{\ lmn} \right\}. \tag{124}$$

By means of (108), (109), and (111), one finds a convenient explicit expression for $t_i^{\ k}$ in the following way. First, we may substitute the usual derivatives $a_{n,\,m}$ in the antisymmetrical expression (108) by covariant derivatives $a_{n;\,m}$ which are tensor components. Hence

$$\mathfrak{F}^{kl} = \frac{\sqrt{-g}}{\varkappa}\left(a^{l;\,k} - a^{k;\,l}\right)$$

and

$$\mathfrak{F}^k = \mathfrak{F}^{kl}{}_{,\,l} = \sqrt{-g}\left[\mathfrak{F}^{kl}/\sqrt{-g}\right]_{;\,l} = \frac{\sqrt{-g}}{\varkappa}\left(a^{l;\,k}{}_{;\,l} - a^{k;\,l}{}_{;\,l}\right). \tag{125}$$

Then, we use the commutation law for repeated covariant derivations:

$$a^{l;\,k}{}_{;\,l} = a^l{}_{;\,i}{}^{;\,k} - R_i^{\ lk}{}_l\, a^i = (a^l_{;\,l})^{;k} - R_i^k\, a^i \tag{126}$$

where R_{iklm} is the Riemann curvature tensor, and R_{ik} is its contraction. By (125), (126), and the field equations (2) we therefore get

$$\mathfrak{F}^k = \sqrt{-g}\, T_i^k\, a^i - \frac{\sqrt{-g}\, R}{2\varkappa} \delta_i^k\, a^i + \frac{\sqrt{-g}}{\varkappa}\left[(a^l_{;\,l})^{;\,k} - (a^{k;\,l})_{;\,l}\right]. \tag{127}$$

Now, choose the a^i equal to constants ε^i; then, according to (111),

$$\mathfrak{F}^k = \varepsilon^i \mathcal{T}_i{}^k = \sqrt{-g}\,(T_i^k + t_i{}^k)\,\varepsilon^i \tag{128}$$

and, by a simple calculation,

$$(a^l_{;\,l})^{;\,k} - (a^{k;\,l})_{;\,l} = \left[(\Gamma^l_{il})^{,\,k} - (\Gamma^k_{il}\,g^{lm})_{,\,m} - \Gamma^k_{in}\,\Gamma^l_{ml}\,g^{mn} - \Gamma^k_{lm}\,\Gamma^m_{in}\,g^{ln}\right]\varepsilon^i. \tag{129}$$

Therefore, since the ε^i are arbitrary, the equations (127), (128) yield

$$t_i{}^k = -\frac{R}{2\,\varkappa}\,\delta_i^k + \hat{t}_i^k \tag{130}$$

with

$$\varkappa\,\hat{t}_i^k = (\Gamma^l_{il})^{,\,k} - (\Gamma^k_{il})^{,\,l} - \Gamma^k_{il}\,(g^{lm}_m + g^{lm}\,\Gamma^n_{mn}) - \Gamma^m_{in}\,\Gamma^k_{lm}\,g^{ln} \tag{131}$$

and

$$(\Gamma^i_{kl})^{,\,m} = (\Gamma^i_{kl})_{,\,n}\,g^{mn}.$$

$\hat{t}_i^k$ differs from $t_i{}^k$ by a tensor. Thus, the transformation law (124) holds also for $\hat{t}_i^k$.

By (10), (13'), and (71) $t_i{}^k$ satisfies the conservation law

$$\left(\sqrt{-g}\,t_i{}^k\right)_{,\,k} = -\left(\sqrt{-g}\,T_i^k\right)_{,\,k} = -\frac{\sqrt{-g}}{2}\,g_{kl,\,i}\,T^{kl} \tag{132}$$

in any system of coordinates. But, as is well known, one may always in an infinite number of ways introduce systems of coordinates which are geodesic at a given point 0 in 4-space, i. e., systems in which the first order derivatives of the metric tensor vanish at the point 0. Then, at 0, which we shall take as origin of the geodesic system, (132) reduces to

$$\left.\begin{aligned} t_i{}^k{}_{,\,k} &= 0 \\ T^k_{i,\,k} &= 0. \end{aligned}\right\} \tag{133}$$

i. e., the conservation law is of the same form as in a system of inertia in the special theory of relativity. Geodesic systems of coordinates are therefore often called *local systems of inertia*. However, the complex $t_i{}^k$ is not in general zero in a geodesic system and, as we shall see in the last section, it seems appropriate to use this denotation only for a certain restricted class of geodesic systems. In a general geodesic system, we have at the origin, according to (129), (131), (113), and (13),

$$\left.\begin{aligned} t_i^{\ k} &= -\delta_i^k R/2\varkappa + \hat{t}_i^k \\ \hat{t}_i^k &= (\Gamma_{il}^l)^{,k} - (\Gamma_{il}^k)^{,l} \\ \chi_i^{\ kl} &= 0, \quad b_i^{\ klm}{}_{,l} = 0. \end{aligned}\right\} \tag{134}$$

Let us now assume that the coordinate systems (x^i) and (x'^i) in (124) are both geodesic at the point 0, which means that the first order derivatives of the $\breve{\alpha}_k^i$ must be zero at 0, i. e.,

$$\breve{\alpha}_{m,\,n}^s(0) = 0. \tag{135}$$

We shall further choose

$$\alpha_k^i(0) = \delta_k^i \tag{135'}$$

which does not imply any essential restriction. In that case, the transformation law (124) at the origin 0 of our geodesic systems takes the form

$$t_i'^{\ k} = t_i^{\ k} - \breve{\alpha}_{i,\,m,\,n}^{r\,(0)}\, b_r^{\ kmn}. \tag{136}$$

Here, $\breve{\alpha}_{i,\,m,\,n}^r$ may be any set of numbers symmetrical in the indices i, m, and n.

The question is now whether the coefficients $\breve{\alpha}_{i,\,m,\,n}^{r\,(0)}$ can be chosen such that the $t_i'^{\ k}$ become zero at 0. It is easily seen that this is not always possible, for the diagonal sum $t_i^{\ i}$ is obviously invariant under the transition from one geodesic system to another. In fact, we get from (136)

$$t_i'^{\ i} = t_i^{\ i}, \tag{137}$$

since the last term in (136) vanishes by contraction of the indices i and k on account of the symmetry of $\breve{\alpha}_{i,\,m,\,n}^r$ and the antisymmetry of $b_r^{\ imn}$ in the indices i and m.

Further, since by (134),

$$\left.\begin{aligned} \hat{t}_i^i &= (\Gamma_{il}^l)^{,i} - (\Gamma_{il}^i)^{,l} = 0 \\ t_i^{\ i} &= -2R/\varkappa, \end{aligned}\right\} \tag{138}$$

it is clear that a transformation (136) cannot make $t_i'^{\ k}$ zero, unless the curvature scalar is zero at the point 0. On the other hand, since $\hat{t}_i'^{\ i} = \hat{t}_i^i = 0$ and the transformation equations (136) hold also for $\hat{t}_i^k$, it seems always possible to choose a geodesic system in which all components $\hat{t}_i'^{\ k}$ vanish.

In the following sections, we shall see that this is really the case for a large group of geodesic systems, the "locally normal" systems of coordinates.

6. Normal Coordinates

Among the coordinate systems which are geodesic at a given point 0, the normal coordinates introduced already by RIEMANN for 2-dimensional surfaces, play a distinguished role. They are defined as follows. Let (x^i) be an arbitrary system of coordinates. Then, the geodesics may be defined by the equations

$$\frac{d^2 x^i}{d\lambda^2} + \Gamma^i_{kl} \frac{dx^k}{d\lambda} \frac{dx^l}{d\lambda} = 0, \tag{139}$$

where the parameter λ is defined only up to a linear transformation. For all geodesics, except the null-lines, λ is proportional to the invariant 4-distance s. Now, consider all the geodesics passing through the point 0. They are defined by the vector tangents at 0 with components $\beta^i = \frac{dx^i}{d\lambda}(0)$. In a certain finite domain around 0, there will be only one of these geodesics passing through a given point P. We may therefore characterize this point by the four numbers

$$\overset{\circ}{x}{}^i = \beta^i (\lambda_P - \lambda_0), \quad \beta^i = \frac{dx^i}{d\lambda}(0) \tag{140}$$

which are the normal coordinates of Riemann. They are uniquely determined since they are unchanged by a linear transformation of the parameter λ. If P approaches 0, the line joining 0 and P defines an infinitesimal vector at 0 with the contravariant components dx^i and $d\overset{\circ}{x}{}^i$ in the two systems of coordinates, respectively. Obviously, we have at 0 $d\overset{\circ}{x}{}^i = dx^i$, i. e., $\alpha^i_k(0) = \delta^i_k$ and the components of any tensor are identical in the two systems at the origin 0. Thus, for instance,

$$\overset{\circ}{g}_{ik}(0) = g_{ik}(0), \quad \overset{\circ}{R}_{iklm}(0) = R_{iklm}(0). \tag{141}$$

In this way, a uniquely defined normal system $\overset{\circ}{x}{}^i$ is connected with every x^i-system. An arbitrary transformation of the x^i-system $(x^i) \to (x'^i)$ obviously induces a *linear* transformation $(\overset{\circ}{x}{}^i) \to (\overset{\circ}{x}{}'^i)$ of the adjoint normal systems. The normal systems of coordinates are as close to the rectilinear systems of flat space as possible in a general Riemannian space. From their definition it follows that any geodesic passing through the origin 0 is described by a linear parameter representation in normal coordinates, i. e.,

$$\overset{\circ}{x}{}^i = \beta^i (\lambda - \lambda_0)$$

with constant β^i. Thus,

$$\frac{d\overset{\circ}{x}^i}{d\lambda} = \beta^i = \overset{\circ}{x}^i/(\lambda - \lambda_0), \quad \frac{d^2\overset{\circ}{x}^i}{d\lambda^2} = 0$$

and, by means of the equations (139) written in normal coordinates, we get

$$\overset{\circ}{\Gamma}{}^i_{kl}(\overset{\circ}{x})\,\overset{\circ}{x}^k\,\overset{\circ}{x}^l = 0 \tag{142}$$

at all points in 4-space. The equations (142) or the equivalent equations

$$\overset{\circ}{\Gamma}_{i,\,kl}(\overset{\circ}{x})\,\overset{\circ}{x}^k\,\overset{\circ}{x}^l = 0 \tag{143}$$

represent a sufficient and necessary condition for the system of coordinates to be a normal system.

By repeated differentiations of these equations, one finds, as shown in Appendix B, the following values for the derivatives of the metric tensor at the origin 0:

$$\left.\begin{aligned}
&\overset{\circ}{g}_{ik,\,l}(0) = 0 && \text{a}\\
&\overset{\circ}{g}_{ik,\,l,\,m}(0) = \overset{\circ}{g}_{lm,\,i,\,k}(0) = -\frac{1}{3}\left[R_{ilmk}(0) + R_{imlk}(0)\right] && \text{b}\\
&\overset{\circ}{g}_{ik,\,l,\,m,\,n}(0) = \frac{1}{3}\left[R_{ilkm;\,n}(0) + R_{imkn;\,l}(0) + R_{inkl;\,m}(0)\right] && \text{c}
\end{aligned}\right\} \tag{144}$$

where $R_{ilmk}(0)$ is the Riemann curvature tensor at 0. In a small surrounding of 0, we have the following approximate expression for $g_{ik}(x)$:

$$\overset{\circ}{g}_{ik}(x) = g_{ik}(0) + \frac{1}{2}\overset{\circ}{g}_{ik,\,l,\,m}(0)\,\overset{\circ}{x}^l\,\overset{\circ}{x}^m + \frac{1}{3!}\overset{\circ}{g}_{ik,\,l,\,m,\,n}(0)\,\overset{\circ}{x}^l\,\overset{\circ}{x}^m\,\overset{\circ}{x}^n \tag{145}$$

with coefficients given by (144). The linear terms are lacking, since a normal system, according to (144a), is a special type of a geodesic system. By means of (12), (13), and (144), we have at the point 0

$$\left.\begin{aligned}
\overset{\circ}{\mathfrak{T}}{}_i^{\,k} &= \frac{\sqrt{-g}}{\varkappa}\left(\overset{\circ}{g}_{in,\,m,\,l} - \overset{\circ}{g}_{im,\,n,\,l}\right) g^{km}\,g^{ln}\\
&= -\frac{\sqrt{-g}}{3\,\varkappa}\left(R_{imln} + R_{ilmn} - R_{inlm} - R_{ilnm}\right) g^{km}\,g^{ln}\\
&= -\frac{\sqrt{-g}}{\varkappa}\,R_{il}{}^{kl} = -\frac{\sqrt{-g}}{\varkappa}\,R_i^k = \sqrt{-g}\;T_i^k - \frac{1}{2}\frac{\sqrt{-g}}{\varkappa}\,R\,\delta_i^k.
\end{aligned}\right\} \tag{146}$$

Here, we have used the symmetry properties of the Riemann tensor and the expression for the contracted curvature tensor.

A comparison of (146) with (10) and (134) gives

$$\left.\begin{aligned} \overset{\circ}{t}{}_i^k &= -\frac{R}{2\varkappa}\delta_i^k \\ \overset{\circ}{\hat{t}}{}_i^k &= 0 \end{aligned}\right\} \quad (147)$$

at the origin of a normal system of coordinates.

Similarly, we get at 0 by (12), (13), and (144)

$$\left.\begin{aligned} \overset{\circ}{\mathcal{T}}{}_{i\;,\,r}^{\;k} &= \frac{\sqrt{-g}}{\varkappa}(\overset{\circ}{g}_{in,\,m,\,l,\,r}-\overset{\circ}{g}_{im,\,n,\,l,\,r})\,g^{km}g^{ln} \\ &= -\frac{\sqrt{-g}}{3\varkappa}(R_{inml;\,r}+R_{ilmr;\,n}+R_{irmn;\,l}-R_{imnl;\,r}-R_{ilnr;\,m}-R_{irnm;\,l})\,g^{km}g^{ln}. \end{aligned}\right\} \quad (148)$$

On account of the symmetry properties of the Riemann tensor and the Bianchi identities, this may be written

$$\overset{\circ}{\mathcal{T}}{}_{i\;,\,r}^{\;k} = -\frac{2\sqrt{-g}}{3\,\kappa}(R_{il}{}^{kl}{}_{;\,r}-R_{ir}{}^{lk}{}_{;\,l}).$$

Further, we have

$$R_{ir}{}^{lk}{}_{;\,l} = R^{lk}{}_{ir;\,l} = -R^{lk}{}_{rl;\,i}-R^{lk}{}_{li;\,r} = R^k_{r;\,i}-R^k_{i;\,r}.$$

Thus, at the origin of a normal system, we have

$$\overset{\circ}{\mathcal{T}}{}_{i\;,\,r}^{\;k} = -\frac{4\sqrt{-g}}{3\,\kappa}(R^k_{i;\,r}-\frac{1}{2}R^k_{r;\,i}). \quad (149)$$

In accordance with (13′), the right-hand side of (149) vanishes on account of the contracted Bianchi identities if we put $r = k$ and sum over k.

7. Locally Normal Coordinates. Local Systems of Inertia in Empty Space

The normal coordinates $(\overset{\circ}{x}{}^i)$ considered in the preceding section are uniquely determined by the conditions (141), (142). Usually, however, one is interested only in systems of coordinates which are *locally* normal, i. e., where (142) is satisfied only approximately in a small region around the origin. In this connection, one can distinguish between locally normal systems of first order, second order, and third order, according as to whether only the first, the first and the second, or all three equations (144), respectively, are satisfied. The locally normal systems of first order are obviously just the geodesic systems. Starting from an arbitrary system of coordinates (x^i), the locally normal systems of n'th order may be obtained by a transformation of the form

$$\overset{\circ}{x}{}^i = P^i_{(n)}(x), \tag{150}$$

where $P^i_{(n)}(x)$ is a polynomial of degree $n+1$ in the coordinate differences $x^i - x^i_0$ with suitable coefficients. Here, (x^i_0) denotes the coordinates of the point 0 in the (x^i)-system. As shown in Appendix C, the locally normally systems of order 2, for instance, are obtained by the transformation

$$\overset{\circ}{x}{}^i = x^i + \frac{1}{2}\Gamma^i_{kl}(0)(x^k - x^k_0)(x^l - x^l_0) + \frac{1}{3!}B^i_{klm}(0)(x^k - x^k_0)(x^l - x^l_0)(x^m - x^m_0) \tag{151}$$

with

$$B^i_{klm} = \frac{1}{3}\left[\Gamma^i_{kl,m} + \Gamma^i_{lm,k} + \Gamma^i_{mk,l} + \Gamma^i_{rk}\Gamma^r_{lm} + \Gamma^i_{rl}\Gamma^r_{mk} + \Gamma^i_{rm}\Gamma^r_{kl}\right]. \tag{152}$$

If we omit the last term in (151) we get the usual transformation leading to the geodesic systems.

Let us now first consider a domain of 4-space where there is no matter present, i. e., where

$$T^k_i = 0, \quad R^k_i = 0, \quad R_{i\ ,r}^{\ k} = 0. \tag{153}$$

In that case, we have, according to (10) and (130),

$$\mathcal{T}_i^{\ k} = \sqrt{-g}\, t_i^{\ k}, \quad t_i^{\ k} = \hat{t}^k_i \tag{154}$$

In empty space, it seems natural to define a local system of inertia as a system in which not only the metric tensor is locally constant to the first order, but in which also the complexes $\mathcal{T}_i^{\ k}$ and $t_i^{\ k}$ vanish at the origin. From this point of view, only normal systems of at least second order should be called local systems of inertia, for only in such systems we have, according to (146), (147), and (153),

$$\overset{\circ}{\mathcal{T}}_i^{\ k}(0) = \overset{\circ}{t}_i^{\ k}(0) = \overset{\circ}{\hat{t}}_i^{\ k}(0) = 0. \tag{155}$$

If the system considered is normal of even higher order, $t_i^{\ k}$ is also *locally constant*, for by (149) and (153)

$$\overset{\circ}{\mathcal{T}}{}_{i\ ,r}^{\ k} = \overset{\circ}{t}{}_{i\ ,r}^{\ k} = 0 \tag{156}$$

at the origin.

On the other hand, inside matter where T_i^k, R_i^k, and in general also R are different from zero, we have by (147) in any normal system of at least second order

$$\overset{\circ}{t}{}_i^{\ k}(0) = -\delta_i^k R(0)/2\varkappa. \tag{157}$$

Thus, the gravitational complex $t_i^{\ k}$ does not vanish, unless $T_i^i = R/\varkappa = 0$ as, for instance, in the case where the matter is a purely electromagnetic field. Moreover, from the considerations in section 5 it follows that it is not possible at all to find a geodesic system in which $t_i^{\ k}$ vanishes exactly. Only the complex $\mathfrak{t}_i^k$ can in general be transformed away completely by introducing locally normal systems of second order. Inside matter, these systems thus hardly deserve the name of local systems of inertia. In a subsequent paper, it will be shown in another connection that it is more natural to reserve this denotation for a class of systems which are only approximately locally normal systems of coordinates.

Appendix A

From (62), (64),

$$\mathfrak{h} = \varkappa h_{r\ ,\,l}^{\ rl} = (\eta g^{lm}_{\ \ m} + 2\,g^{lm}\,\eta_m)_{,\,l} \tag{A 1}$$

with

$$\eta = \sqrt{-g}, \qquad \eta_m = \frac{\partial\,\eta}{\partial\,x^m},$$

we get, for arbitrary variations of g^{ik},

$$\delta\,\mathfrak{h} = \eta\,\delta\,g^{lm}_{l,\,m} + g^{rs}_{r,\,s}\,\delta\eta + 3\,g^{kr}_r\,\delta\eta_k + 3\,\eta_l\,\delta\,g^{lm}_m + 2\,\eta_{l,\,m}\,\delta\,g^{lm} + 2\,g^{ik}\,\delta\eta_{i,\,k}\,. \tag{A 2}$$

Differentiation of the relation

$$\delta\eta = -\frac{1}{2}\,\eta\,g_{lm}\,\delta\,g^{lm} \tag{A 3}$$

yields

$$\delta\eta_k = -\frac{1}{2}\,[(\eta\,g_{lm})_{,\,k}\,\delta\,g^{lm} + \eta\,g_{lm}\,\delta\,g^{lm}_k] \tag{A 4}$$

$$\delta\eta_{i,\,k} = -\frac{1}{2}\left[(\eta\, g_{lm})_{,\,i,\,k}\,\delta g^{lm} + (\eta\, g_{lm})_{,\,k}\,\delta g_i^{lm} + (\eta\, g_{lm})_{,\,i}\,\delta g_k^{lm} + \eta\, g_{lm}\,\delta g_{i,\,k}^{lm}\right]. \qquad \text{(A 5)}$$

After introduction of these expressions into (A 2) and some rearrangement of terms, $\delta\,\mathfrak{h}$ may be written

$$\left.\begin{aligned} \delta\,\mathfrak{h} &= \eta\left[\frac{1}{2}\left(\delta_l^i\,\delta_m^k + \delta_l^k\,\delta_m^i\right) - g^{ik}\,g_{lm}\right]\delta g_{i,\,k}^{lm} \\ &+ \left[\frac{3}{2}\left(\eta_l\,\delta_m^k + \delta_l^k\,\eta_m\right) - \frac{3}{2}\,g_r^{kr}\,\eta\,g_{lm} - 2\,g^{rk}\,\eta\,g_{lm}\right)_{,\,r}\Big]\,\delta g_k^{lm} \\ &+ \left[2\,\eta_{l,\,m} - g^{rs}\,(\eta\,g_{lm})_{,\,r,\,s} - \frac{3}{2}\,g_s^{rs}\,(\eta\,g_{lm})_{,\,r} - \frac{1}{2}\,\eta\,g_{lm}\,g_{r,\,s}^{rs}\right]\delta g^{lm}. \end{aligned}\right\} \qquad \text{(A 6)}$$

Hence, by definition,

$$\frac{\partial\,\mathfrak{h}}{\partial\, g^{lm}} = 2\,\eta_{l,\,m} - g^{rs}\,(\eta\,g_{lm})_{,\,r,\,s} - \frac{3}{2}\,(\eta\,g_{lm})_{,\,r}\,g_s^{rs} - \frac{1}{2}\,\eta\,g_{lm}\,g_{r,\,s}^{rs} \qquad \text{(A 7)}$$

$$\frac{\partial\,\mathfrak{h}}{\partial\, g_k^{lm}} = \frac{3}{2}\left(\delta_l^k\,\eta_m + \eta_l\,\delta_m^k\right) - \frac{3}{2}\,g_r^{kr}\,\eta\,g_{lm} - 2\,g^{kr}\,(\eta\,g_{lm})_{,\,r} \qquad \text{(A 8)}$$

$$\frac{\partial\,\mathfrak{h}}{\partial\, g_{i,\,k}^{lm}} = \eta\left[\frac{1}{2}\left(\delta_l^i\,\delta_m^k + \delta_l^k\,\delta_m^i\right) - g^{ik}\,g_{lm}\right]. \qquad \text{(A 9)}$$

These expressions are easily seen to be in accordance with the identities (66).

From (A 9) we get, using the relation

$$\eta_i = -\frac{1}{2}\,\eta\,g_{lm}\,g_i^{lm}, \qquad \text{(A 10)}$$

$$\left(\frac{\partial\,\mathfrak{h}}{\partial\, g_{i,\,k}^{lm}}\,g_i^{lm}\right)_{,\,k} = \left[\eta\,g_i^{ik} - \eta\,g^{ik}\,g_{lm}\,g_i^{lm}\right]_{,\,k} = \left[\eta\,g_i^{ik} + 2\,g^{ik}\,\eta_i\right]_{,\,k} = \mathfrak{h} \qquad \text{(A 11)}$$

which shows that the last term in (74) is zero. Further, by (A 8) and (A 9),

$$\left.\begin{aligned} \frac{\partial\,\mathfrak{h}}{\partial\, g_k^{lm}} - \left(\frac{\partial\,\mathfrak{h}}{\partial\, g_{k,r}^{lm}}\right)_{,\,r} &= \delta_l^k\,\eta_m + \eta_l\,\delta_m^k - \frac{1}{2}\,g_r^{kr}\,\eta\,g_{lm} - g^{kr}\,(\eta\,g_{lm})_{,\,r}, \\ \left[\frac{\partial\,\mathfrak{h}}{\partial\, g_k^{lm}} - \left(\frac{\partial\,\mathfrak{h}}{\partial\, g_{k,\,r}^{lm}}\right)_{,\,r}\right] g_i^{lm} &= 2\,g_i^{kl}\,\eta_l + g_r^{kr}\,\eta_i - g^{kr}\,(\eta\,g_{lm})_{,\,r}\,g_i^{lm}, \end{aligned}\right\} \qquad \text{(A 12)}$$

$$\left.\begin{aligned}\left(\frac{\partial \mathfrak{h}}{\partial g^{lm}_{k,r}}\right)_{,i} g^{lm}_r &= \left[\eta_i \frac{1}{2}(\delta^r_l \delta^k_m + \delta^k_l \delta^r_m) - g^{kr}_i \eta\, g_{lm} - g^{kr}(\eta\, g_{lm})_{,i}\right] g^{lm}_r \\ &= g^{kr}_r \eta_i + 2\, g^{kr}_i \eta_r - g^{kr}(\eta\, g_{lm})_{,i}\, g^{lm}_r .\end{aligned}\right\} \quad \text{(A 13)}$$

Thus, we get for the quantity A^k_i of (74),

$$A^k_i = g^{kr}\left[-\eta_r g_{lm} g^{lm}_i + \eta_i g_{lm} g^{lm}_r - \eta\,(g_{lm,r}\, g^{lm}_i - g_{lm,i}\, g^{lm}_r)\right] = 0, \quad \text{(A 14)}$$

on account of (A 10) and the relation

$$g_{lm,i} = -g_{ls}\, g^{st}_i\, g_{tm} \quad \text{(A 15)}$$

which follows from the equation

$$g_{il}\, g^{kl} = \delta^k_i \quad \text{(A 16)}$$

by differentiation.

Further, we have

$$\left.\begin{aligned}\frac{\partial \mathfrak{h}}{\partial g^{rs}_{l,m}} g^{rs}_m &= \left[\frac{\eta}{2}(\delta^l_r \delta^m_s + \delta^m_r \delta^l_s) - \eta\, g^{lm} g_{rs}\right] g^{rs}_m \\ &= \eta\, g^{lm}_m + 2\, g^{lm} \eta_m = \varkappa\, h^{rl}_r\end{aligned}\right\} \quad \text{(A 17)}$$

which by (75) leads to the expression (76) for K^{kl}_i.

Finally, we have to calculate the quantity B^{kl}_i defined by (81) in the text. For the first term we get by (A 12)

$$\left[\frac{\partial \mathfrak{h}}{\partial g^{in}_l} - \left(\frac{\partial \mathfrak{h}}{\partial g^{in}_{l,m}}\right)_{,m}\right] g^{kn} = \left[\delta^l_i \eta_n + \eta_i \delta^l_n - \frac{1}{2} g^{lr}_r \eta\, g_{in} - g^{lr}(\eta\, g_{in})_{,r}\right] g^{kn}$$

$$= \delta^l_i\, g^{kn} \eta_n + \eta_i\, g^{kl} - \frac{1}{2}\delta^k_i \eta\, g^{lr}_r - g^{lm}(\eta_m g_{in} + \eta\, g_{in,m})\, g^{kn}$$

for the second and third terms by (A 9)

$$\frac{\partial \mathfrak{h}}{\partial g^{in}_{l,m}} g^{kn}_m = \left[\frac{1}{2}\eta\,(\delta^l_i \delta^m_n + \delta^m_i \delta^l_n) - \eta\, g^{lm} g_{in}\right] g^{kn}_m = \frac{\eta}{2}\delta^l_i\, g^{km}_m + \frac{\eta}{2} g^{kl}_i + g^{lm}\eta\, g_{in,m}\, g^{kn},$$

$$-\frac{1}{3}\left[\left(\frac{\partial \mathfrak{h}}{\partial g^{in}_{l,m}}\right) g^{kn}\right]_{,m} = -\frac{1}{3}\left[\frac{\eta}{2}(\delta^l_i \delta^m_n + \delta^m_i \delta^l_n)\, g^{kn} - g^{lm}\eta\, g_{in}\, g^{kn}\right]_{,m}$$

$$= -\frac{1}{6}\delta^l_i (\eta\, g^{km})_{,m} - \frac{1}{6}(\eta\, g^{kl})_{,i} + \frac{1}{3}\delta^k_i (\eta\, g^{lm})_{,m}.$$

Hence,

$$\left.\begin{aligned}\varkappa B_i^{kl} = \frac{1}{2}\delta_i^l(\eta\, g_m^{km} + 2\, g^{km}\eta_m) - \frac{1}{2}\delta_i^k(\eta\, g_m^{lm} + 2\, g^{lm}\eta_m) + \frac{1}{3}\delta_i^k(\eta\, g^{lm})_{,\,m}\\ - \frac{1}{6}\delta_i^l(\eta\, g^{km})_{,\,m} + \eta_i\, g^{kl} + \frac{\eta}{2} g_i^{kl} - \frac{1}{6}(\eta\, g^{kl})_{,\,i}\end{aligned}\right\} \quad (A\,18)$$

and

$$B_i^{kl} - B_i^{lk} = \delta_i^l\, h_r{}^{rk} - \delta_i^k\, h_l{}^{rl} + \frac{1}{2\varkappa}[\delta_i^k(\eta\, g^{lm}) - \delta_i^l(\eta\, g^{km})]_{,\,m}. \quad (A\,19)$$

On the other hand, we have by (6), and (7) in the text

$$s_i{}^{kl} - s_i{}^{lk} = 2\, h_i{}^{kl} + \frac{1}{2\varkappa}[\delta_i^l(\eta\, g^{km}) - \delta_i^k(\eta\, g^{lm})]_{,\,m}$$

$$2\, h_i{}^{kl} = \frac{\sqrt{-g}}{\varkappa} g_{in}(g_m^{kn}\, g^{lm} - g_m^{ln}\, g^{km}) + \frac{1}{\varkappa}\left[\delta_i^k \frac{1}{\eta}(\eta^2\, g^{lm})_{,\,m} - \delta_i^l \frac{1}{\eta}(\eta^2\, g^{km})_{,\,m}\right].$$

Hence, by (64), (80),

$$\left.\begin{aligned}U_i{}^{kl} = s_i{}^{kl} - s_i{}^{lk} + B_i^{kl} - B_i^{lk} = \frac{\sqrt{-g}}{\varkappa} g_{in}(g_m^{kn}\, g^{lm} - g_m^{ln}\, g^{km})\\ = \frac{\sqrt{-g}}{\varkappa}(g_{in,\,m} - g_{im,\,n})\, g^{km}\, g^{ln} = \chi_i^{kl},\end{aligned}\right\} \quad (A\,20)$$

where $\chi_i{}^{kl}$ is given by equation (13) in the text.

Appendix B

A system of normal Riemann coordinates in 4-space is characterized by the equations (143) which have to be satisfied at each point. Hence, omitting the ° over the symbols, thus writing x^i, g_{ik}, . . . instead of $\overset{\circ}{x}{}^i$, $\overset{\circ}{g}_{ik}$, . . .,

$$\Gamma_{i,\,rs}(x)\, x^r\, x^s = 0. \quad (B\,1)$$

If we differentiate this equation twice with respect to x^k and x^l, and hereafter with respect to x^m, we get the following two equations:

$$(\Gamma_{i,\,rs})_{,\,k,\,l}(x)\, x^r\, x^s + 2\,[(\Gamma_{i,\,kr})_{,\,l} + (\Gamma_{i,\,lr})_{,\,k}]\, x^r + 2\,\Gamma_{i,\,kl}(x) = 0 \quad (B\,2)$$

$$\left.\begin{aligned}(\Gamma_{i,rs})_{,k,l,m}(x)\,x^r x^s + 2\,[(\Gamma_{i,mr})_{,k,l} + (\Gamma_{i,kr})_{,l,m} + (\Gamma_{i,lr})_{,k,m}]\,x^r \\ + 2\,[(\Gamma_{i,km})_{,l} + (\Gamma_{i,lm})_{,k} + (\Gamma_{i,kl})_{,m}] = 0\,.\end{aligned}\right\} \qquad \text{(B 3)}$$

For the values of $\Gamma_{i,kl}$ and $(\Gamma_{i,kl})_{,m}$ at the origin 0, we thus get, by putting $x^i = 0$ in these equations,

$$\Gamma_{i,kl} = 0 \qquad \text{(B 4)}$$

$$A_{iklm} \equiv \underset{(klm)}{\mathsf{S}}\,(\Gamma_{i,kl})_{,m} \equiv (\Gamma_{i,kl})_{,m} + (\Gamma_{i,lm})_{,k} + (\Gamma_{i,mk})_{,l} = 0\,. \qquad \text{(B 5)}$$

Here, as in the following, the symbol $\underset{(klm)}{\mathsf{S}}$ in front of a term containing the indices k, l, and m means addition of the two terms obtained by cyclic permutation of these indices.

Similarly, one finds, by differentiation of (B 3) with respect to x^n and afterwards putting $x^i = 0$, at the point 0 the relation

$$A_{ilmnk} + B_{iklmn} = 0 \qquad \text{(B 6)}$$

with

$$\left.\begin{aligned} A_{ilmnk} &= \underset{(lmn)}{\mathsf{S}}\,(\Gamma_{i,lm})_{,n,k} \\ B_{iklmn} &= \underset{(lmn)}{\mathsf{S}}\,(\Gamma_{i,kl})_{,m,n}\,. \end{aligned}\right\} \qquad \text{(B 7)}$$

The equation (B 4) is equivalent to

$$g_{ik,l} = 0 \qquad \text{(B 8)}$$

showing that the system is geodesic at 0.

The Christoffel symbols are defined by

$$\Gamma_{i,kl} = \frac{1}{2}\,(g_{ik,l} + g_{il,k} - g_{kl,i})\,. \qquad \text{(B 9)}$$

Introduction of these expressions into (B 5) gives

$$A_{iklm} \equiv \underset{(klm)}{\mathsf{S}}\, g_{ik,l,m} - \frac{1}{2}\,\underset{(klm)}{\mathsf{S}}\, g_{kl,m,i} = 0\,,$$

which is equivalent to

$$2\,A_{iklm} + A_{kilm} \equiv 3\,g_{ik,l,m} + \frac{3}{2}\,(g_{il,k,m} + g_{im,k,l} - g_{lm,i,k}) = 0\,.$$

Hence, since $g_{ik,\,l,\,m}$ is symmetrical in i and k,

$$g_{ik,\,l,\,m} = -(\Gamma_{i,\,lm})_{,\,k} = -(\Gamma_{k,\,lm})_{,\,i}. \tag{B 10}$$

Similarly, we get for A_{ilmnk} and B_{iklmn} in (B 7), after a simple rearrangement of terms,

$$A_{ilmnk} = \underset{(lmn)}{\mathsf{S}}\, g_{il,\,m,\,n,\,k} - \frac{1}{2} \underset{(lmn)}{\mathsf{S}}\, g_{lm,\,n,\,i,\,k} \tag{B 11}$$

$$B_{iklmn} = \frac{3}{2} g_{ik,\,l,\,m,\,n} + \frac{1}{2} \underset{(lmn)}{\mathsf{S}} \left[g_{il,\,m,\,n,\,k} - g_{kl,\,m,\,n,\,i} \right]. \tag{B 12}$$

Since the last term on the right-hand side of (B 11), as well as the first term on the right-hand side of (B 12), is symmetrical in i and k, we get

$$B_{iklmn} - B_{kilmn} = \underset{(lmn)}{\mathsf{S}} \left[g_{il,\,m,\,n,\,k} - g_{kl,\,m,\,n,\,i} \right] = A_{ilmnk} - A_{klmni}. \tag{B 13}$$

On the other hand, we have by (B 6)

$$B_{iklmn} - B_{kilmn} = -(A_{ilmnk} - A_{klmni}),$$

which means that these differences must be zero. Hence, by (B 12) and (B 6),

$$\left.\begin{aligned} g_{ik,\,l,\,m,\,n} &= \frac{2}{3} B_{iklmn} = -\frac{2}{3} A_{ilmnk} \\ g_{ik,\,l,\,m,\,n} &= -\frac{2}{3} \underset{(lmn)}{\mathsf{S}} (\Gamma_{i,\,lm})_{,\,n,\,k} = -\frac{2}{3} \left[(\Gamma_{i,\,lm})_{,\,n} + (\Gamma_{i,\,mn})_{,\,l} + \Gamma_{i,\,nl})_{,\,m} \right]_{,\,k}. \end{aligned}\right\} \tag{B 14}$$

At the origin 0, where (B 4) and (B 8) hold, the Riemann curvature tensor R_{iklm} and its derivatives are now given by

$$\left.\begin{aligned} R_{iklm} &= (\Gamma_{i,\,kl})_{,\,m} - (\Gamma_{i,\,km})_{,\,l} \\ R_{iklm;\,n} = R_{iklm,\,n} &= (\Gamma_{i,\,kl})_{,\,m,\,n} - (\Gamma_{i,\,km})_{,\,l,\,n}. \end{aligned}\right\} \tag{B 15}$$

Hence, by (B 5), (B 10),

$$R_{ilmk} + R_{imlk} = 2(\Gamma_{i,\,lm})_{,\,k} - (\Gamma_{i,\,kl})_{,\,m} - (\Gamma_{i,\,mk})_{,\,l} = 3(\Gamma_{i,\,lm})_{,\,k}$$

and

$$g_{ik,\,l,\,m} = -\frac{1}{3}(R_{ilmk} + R_{imlk}). \tag{B 16}$$

Similarly, by (B 15), (B 6), (B 7) and (B 14)

$$\underset{(lmn)}{\mathsf{S}} R_{ilkm;\,n} = \underset{(lmn)}{\mathsf{S}} [(\Gamma_{i,\,kl})_{,\,m,\,n} - (\Gamma_{i,\,lm})_{,\,n,\,k}] = -2 \underset{(lmn)}{\mathsf{S}} (\Gamma_{i,\,lm})_{,\,n,\,k}$$

and

$$g_{ik,\,l,\,m,\,n} = \frac{1}{3} \underset{(lmn)}{\mathsf{S}} R_{ilkm;\,n} = \frac{1}{3} [R_{ilkm;\,n} + R_{imkn;\,l} + R_{inkl;\,m}]. \qquad \text{(B 17)}$$

The equations (B 8), (B 16), and (B 17) are just the equations (144) used in Section 6.

Appendix C

We shall in this Appendix consider the transformations leading from an arbitrary system of coordinates x^i to a normal system $\overset{\circ}{x}{}^i$ with origin at a given point 0. Put

$$\left.\begin{aligned} \overset{\circ}{x}{}^i &= f^i(x), \quad \alpha^i_k = \frac{\partial \overset{\circ}{x}{}^i}{\partial x^k}, \quad \alpha^i_{k,\,l} = \frac{\partial^2 \overset{\circ}{x}{}^i}{\partial x^k \, \partial x^l}, \; \ldots \\ x^i &= g^i(\overset{\circ}{x}), \quad \breve{\alpha}^i_k = \frac{\partial x^i}{\partial \overset{\circ}{x}{}^k}, \quad \breve{\alpha}^i_{k,\,l} = \frac{\partial^2 x^i}{\partial \overset{\circ}{x}{}^k \, \partial \overset{\circ}{x}{}^l}, \; \ldots \end{aligned}\right\} \qquad \text{(C 1)}$$

$$\alpha^i_l \, \breve{\alpha}^l_k = \breve{\alpha}^i_l \, \alpha^l_k = \delta^i_k. \qquad \text{(C 2)}$$

By means of (C 2), the Christoffel transformation formulae

$$\overset{\circ}{\Gamma}{}^i_{kl}(\overset{\circ}{x}) = \alpha^i_r \, \breve{\alpha}^r_{k,\,l} + \alpha^i_r \, \breve{\alpha}^s_k \, \breve{\alpha}^t_l \, \Gamma^r_{st}(x)$$

may be written

$$\breve{\alpha}^i_{k,\,l} + \Gamma^i_{st}(x^m) \, \breve{\alpha}^s_k \, \breve{\alpha}^t_l = \breve{\alpha}^i_r \, \overset{\circ}{\Gamma}{}^r_{kl}(\overset{\circ}{x}). \qquad \text{(C 3)}$$

If we multiply this equation by $\overset{\circ}{x}{}^k \, \overset{\circ}{x}{}^l$, we get by Eq. (143) in the text the following differentio-functional equations for the functions $g^i(\overset{\circ}{x})$

$$\overset{\circ}{x}{}^k \, \overset{\circ}{x}{}^l \frac{\partial^2 g^i}{\partial \overset{\circ}{x}{}^k \, \partial \overset{\circ}{x}{}^l} + \Gamma^i_{st}(g^m(\overset{\circ}{x})) \frac{\partial g^s}{\partial \overset{\circ}{x}{}^k} \frac{\partial g^t}{\partial \overset{\circ}{x}{}^l} = 0. \qquad \text{(C 4)}$$

Four independent solutions of (C 4) satisfying the condition

$$g^i(0) = x^i_0, \quad \frac{\partial g^i}{\partial \overset{\circ}{x}{}^k}(0) = \delta^i_k$$

define the transformation to normal coordinates.

If we are interested in locally normal systems only, we need not consider the values of $g^i(\overset{\circ}{x})$ in large distances from 0. From the inverse relation (C 3), i. e.,

$$\alpha^i_{k,l}(x) + \overset{\circ}{\Gamma}{}^i_{st}(\overset{\circ}{x})\,\alpha^s_k(x)\,\alpha^t_l(x) = \alpha^i_r(x)\,\Gamma^r_{kl}(x) \tag{C 5}$$

we get at 0, remembering that $\alpha^i_k(0) = \delta^i_k$ and $\overset{\circ}{\Gamma}{}^i_{kl}(0) = 0$,

$$\alpha^i_{k,l}(0) = \Gamma^i_{kl}(0) \tag{C 6}$$

which, apart from a factor 2 is identical with the coefficient of the quadratic term in the transformation (151).

Now, differentiate (C 5) with respect to x^m:

$$\left.\begin{aligned}\alpha^i_{k,l,m}(x) + \overset{\circ}{\Gamma}{}^i_{st,r}(\overset{\circ}{x})\,\alpha^r_m\,\alpha^s_k\,\alpha^t_l + \overset{\circ}{\Gamma}{}^i_{st}(\overset{\circ}{x})\,(\alpha^s_{k,m}\,\alpha^t_l + \alpha^s_k\,\alpha^t_{l,m})\\ = \alpha^i_{r,m}\,\Gamma^r_{kl}(x) + \alpha^i_r\,\Gamma^r_{kl,m}(x).\end{aligned}\right\} \tag{C 7}$$

At 0, this gives

$$\alpha^i_{k,l,m}(0) + \overset{\circ}{\Gamma}{}^i_{kl,m}(0) = \alpha^i_{r,m}(0)\,\Gamma^r_{kl}(0) + \Gamma^i_{kl,m}(0). \tag{C 8}$$

Thus, since $\alpha^i_{k,l,m}$ is symmetrical in the indices k, l, m, we get, by (C 6) and the equation

$$\overset{\circ}{\Gamma}{}^i_{kl,m}(0) + \overset{\circ}{\Gamma}{}^i_{lm,k}(0) + \overset{\circ}{\Gamma}{}^i_{mk,l}(0) = 0 \tag{C 9}$$

valid in a locally normal system (see (144 b) and (B 5) in Appendix B),

$$\alpha^i_{k,l,m}(0) = \frac{1}{3}\underset{(klm)}{\mathsf{S}}\left[\Gamma^i_{kl,m}(0) + \Gamma^i_{rm}(0)\,\Gamma^r_{kl}(0)\right] \tag{C 10}$$

which is identical with the coefficient $B^i_{klm}(0)$ in the transformation (151). Thus, the latter transformation leads to a locally normal system of the second order. Proceeding in this way, we can by further differentiations of (C 7) and subsequently putting $x^i = 0$ derive expressions for the values of still higher derivatives of α^i_k at 0 and, thus, determine the coefficients in the higher order terms in the polynomial $P^i_{(n)}(x)$ of Eq. (150) which defines the transformations to the locally normal systems of arbitrarily high order.

Institut for Teoretisk Fysik and
NORDITA – Nordisk Institut for Teoretisk Atomfysik,
København, Danmark.

References and Notes

(1) See, for instance, a paper by J. N. Goldberg, Phys. Rev. **111**, 315 (1958) and by P. G. Bergmann, Phys. Rev. **112**, 287 (1958).
(2) A. Einstein, Berlin. Ber. p. 167 (1916); R. C. Tolman Phys. Rev. **35**, 875 (1930).
(3) See, for instance, the Appendix of reference 4. After this paper was finished I noticed that the existence of the antisymmetric quantity χ_i^{kl}, which has been given the name of super potentials, was established many years ago, first in a paper by Ph. von Freund, Am. Math. **40**, 417 (1939), later by H. Zatzkis, Phys. Rev. **81**, 1023 (1951); see also P. G. Bergmann Phys. Rev. **75**, 680 (1949); P. G, Bergmann and R. Schiller, Phys. Rev. **89**, 4 (1953), and J. N. Goldberg, Phys. Rev. **89**, 263 (1953), **99**, 1873 (1955).
(4) C. Møller, Annals of Physics **4**, 347 (1958).
(5) C. Møller, Über die Energie Nichtabgeschlossener Systeme in der Allgemeinen Relativitätstheorie. Max Planck Festschrift, Artikel 9, Deutscher Verlag der Wissenschaften, Berlin (1958); see also a forthcoming paper by M. Magnusson.
(6) L. Rosenfeld, Acad. Roy. Belg. Mémoires XVIII, Fasc. 6 (1940); F. Belinfante, Physica **6**, 887 (1939).
(7) For a survey of these problems, see P. G. Bergmann, Irwin Goldberg, Allen Janis, and Esra Newman, Phys. Rev. **103**, 807 (1956); P. G. Bergmann and R. Schiller, Phys. Rev. **89**, 4 (1953); Bryce S. DeWitt, Revs. Mod. Phys. **29**, 377 (1957); P. A. M. Dirac, Proc. Roy. Soc. A **246**, 333 (1958).
(8) A. Komar, Covariant Conservation Laws in General Relativity. Phys. Rev. **113**, 934 (1959).

Indleveret til selskabet den 23. januar 1959.
Færdig fra trykkeriet den 24. juni 1959.